# 3. Geometric Notions

*At the basis of the distance concept lies, for example, the concept of convergent point sequence and their defined limits, and one can, by choosing these ideas as those fundamental to point set theory, eliminate the notions of distance.*

Felix Hausdorff

By choosing open sets as the basic notion we next generalize familiar analytic and geometric notions from Euclidean space to this new setting. Two fundamental notions were introduced by Georg Cantor (1845–1918) in his work [?] on analysis:

Definition 3.1. *Let $(X, \mathcal{T})$ be a topological space. A subset $K$ of $X$ is* **closed** *if its complement in $X$ is open. If $A \subseteq X$, a topological space and $x \in X$, then $x$ is a* **limit point** *of $A$, if, whenever $U \subset X$ is open and $x \in U$, then there is some $y \in U \cap A$, with $y \neq x$.*

Closed sets are the natural generalization of closed sets in $\mathbb{R}^n$. Notice that an arbitrary subset of a topological space can be neither open nor closed, for example, $[a, b) \subset \mathbb{R}$ in the usual topology. A slogan to remember is that *"a subset is not a door."*

In a metric space the notion of a limit point $w$ of a subset $A$ is given by a sequence $\{x_i, i = 1, 2, \ldots\}$ with $x_i \in A$ for all $i$ and $\lim_{i\to\infty} x_i = w$. The limit means, for any $\epsilon > 0$, there is an integer $N$ for which whenever $n \geq N$, we have $d(x_n, w) < \epsilon$. We distinguish two cases: If $w \in A$, then a constant sequence will converge to $w$, and we want, for each $\epsilon > 0$, that there be some other point $a_\epsilon \in A$ with $a_\epsilon \neq w$ and $a_\epsilon \in B(w, \epsilon)$. Notice that, if we take $x_i = a_{1/i}$, then $\lim_{i\to\infty} x_i = w$ follows. Conversely, if there is a sequence of infinitely many distinct points $x_i \in A$ with $\lim_{i\to\infty} x_i = w$, then $w$ is a limit point of $A$.

In the most general topological spaces, the situation can be quite different. The limit points of a subset of a metric space are "near" the subset. Consider $\mathbb{R}$ with the finite-complement topology and let $A = \mathbb{Z}$, the set of integers in $\mathbb{R}$. Choose any real number $r$ and suppose $U$ is an open set containing $r$. Then $U = \mathbb{R} - \{s_1, s_2, \ldots, s_k\}$ for some choices of real numbers $s_1, \ldots, s_k$. Since this set leaves out only finitely many points and $\mathbb{Z}$ is infinite, there are infinitely many integers in $U$ and certainly one not equal to $r$. Thus $r$ is a limit point of $\mathbb{Z}$. This is an extreme case—every point in the space is a limit point of a proper subset.

Closed sets and limit points are related.

Proposition 3.2. *A subset $K$ of a topological space $(X, \mathcal{T})$ is closed if and only if it contains all of its limit points.*

*Proof:* Suppose $K$ is closed, $x \in X$ is some point, and $x \notin K$. Then $x \in X - K$ and $X - K$ is open. So $x$ is contained in an open set that does not intersect $K$, and therefore, $x$ is not a limit point of $K$. Thus all limit points of $K$ must be in $K$.

Suppose $K$ contains all of its limit points. Let $x \in X - K$, then $x$ is not a limit point and so there exists an open set $U^x$ with $x \in U^x$ and $U^x \cap K = \emptyset$, that is, $U^x \subset X - K$. Since we can find such an open set $U^x$ for all $x \in X - K$, we have

$$X - K \subset \bigcup_{x \in X - K} U^x \subset X - K.$$

We have written $X - K$ a a union of open sets. Hence $X - K$ is open and $K$ is closed. $\diamondsuit$

Let $(X, \mathcal{T})$ be a topological space and $A$ an arbitrary subset of $X$. We associate to $A$ subsets definable with the open sets in the topology as follows:

DEFINITION 3.3. *The* **interior** *of $A$ is the largest open set contained in $A$, that is,*

$$\text{int } A = \bigcup\nolimits_{U \subseteq A, \text{ open}} U.$$

*The* **closure** *of $A$ is the smallest closed set in $X$ containing $A$, that is,*

$$\text{cls } A = \bigcap\nolimits_{K \supseteq A, \text{ closed}} K.$$

These operations tell us something geometric about subsets, for example, the subset $\mathbb{Q} \subset (\mathbb{R}, \text{usual})$ has empty interior and closure all of $\mathbb{R}$. To see this suppose $U \subset \mathbb{R}$ is open. Then there is an interval $(a, b) \subset \mathbb{R}$ for some $a < b$. Since $(a, b) \cap \mathbb{R} - \mathbb{Q} \neq \emptyset$, $U \not\subset \mathbb{Q}$ and so int $\mathbb{Q} = \emptyset$. If $\mathbb{Q} \subset K$ is a closed subset of $\mathbb{R}$, then $\mathbb{R} - K$ is open and contains no rationals. It follows that it contains no interval because every nonempty interval of real numbers contains a rational number. Thus $\mathbb{R} - K = \emptyset$ and cls $\mathbb{Q} = \mathbb{R}$.

The operation of closure ought to be a kind of 'closing' up of the set by putting in all the 'ragged edges.' We make this precise as follows:

PROPOSITION 3.4. *If $A \subset X$, a topological space, then* cls $A = A \cup A'$ *where*

$$A' = \{ \text{ limit points of } A \ \}.$$

*$A'$ is called the* **derived set** *of $A$.*

*Proof:* By definition, cls $A$ is closed and contains $A$ so $A \subset$ cls $A$. It follows that if $x \notin$ cls $A$, then there exists an open set $U$ containing $x$ with $U \cap A = \emptyset$ and so $x \notin A$ and $x \notin A'$. This shows $A \cup A' \subset$ cls $A$. To show the other containment, suppose $y \in$ cls $A$ and $V$ is an open set containing $y$. If $V \cap A = \emptyset$, then $A \subset (X - V)$ a closed set and so cls $A \subset (X - V)$. But then $y \notin$ cls $A$, a contradiction. If $y \in$ cls $A$ and $y \notin A$, then, for any open set $V$ with $y \in V$, we have $V \cap A \neq \emptyset$ and so $y$ is a limit point of $A$. Thus cls $A \subset A \cup A'$. $\diamondsuit$

With these definitions we have the following sequence of subsets:

$$\text{int } A \subset A \subset \text{cls } A = A \cup A'.$$

We add another more refined distinction between points in the closure.

DEFINITION 3.5. *Let $A$ be a subset of $X$, a topological space. A point $x \in X$ is in the* **boundary** *of $A$, if for any open set $U \subset X$ with $x \in U$, we have $U \cap A \neq \emptyset$ and $U \cap (X - A) \neq \emptyset$. The set of points in the boundary of $A$ is denoted* bdy$A$.

A boundary point of a subset is "on the edge" of the set. For example, suppose $A = (0, 1] \cup \{2\}$ in $\mathbb{R}$ with the usual topology. The point 0 is a boundary point, a point in the derived set, but not in $A$; 1 is a boundary point, a point in the derived set, and a point in $A$; and 2 is boundary point, not in the derived set, but in $A$.

The boundary points lie outside the interior of $A$. We next see how the boundary relates to the closure.

PROPOSITION 3.6. cls $A = \text{int } A \cup \text{bdy}A$.

*Proof:* Suppose $x \in \text{bdy}A$ and $K \subset X$ is closed with $A \subset K$. If $x \notin K$, then the open set $V = X - K$ contains $x$. Since $x \in \text{bdy}A$, we have $V \cap A \neq \emptyset \neq V \cap (X - A)$. But $A \subset K$ implies $V \cap A = \emptyset$, a contradiction. Thus $\text{bdy}A \subset \text{cls } A$, and so $\text{bdy}A \cup \text{int } A \subset \text{cls } A$.

We have already shown that $A \cup A' = \text{cls } A$. If $x \in A - \text{int } A$, then for any open set $U$ containing $x$, $U \cap (X - A) \neq \emptyset$, otherwise $x$ would be in the interior of $A$. By virtue of $x \in A$, $U \cap A \neq \emptyset$, so $x \in \text{bdy}A$. Thus $\text{int } A \cup \text{bdy}A \supset A$. Consider $y \in A' - A$ and any open set $V$ containing $y$. Since $y \in A'$, $V \cap A \neq \emptyset$. Also $V \cap (X - A) \neq \emptyset$ since $y \notin A$. Thus $A'$ is a subset of $\text{bdy}A$ and $\text{cls } A \subset \text{int } A \cup \text{bdy}A$. ♢

Suppose $(X, d)$ is a metric space. Then limit points agree with our usual intuitive idea. A sequence $\{x_n\}$ in $X$ *converges* to a point $x \in X$, if for any $\epsilon > 0$ there is a natural number $N = N(\epsilon)$ so that $x_n \in B(x, \epsilon)$ whenever $n \geq N$. Consider a subset $A \subset X$ and suppose $x \in A' - A = A' \cap (X - A)$. For each $n = 1, 2, 3, \ldots$ there is the open set $B(x, 1/n)$. Since $x$ is a limit point, $B(x, 1/n) \cap A \neq \emptyset$. Let $x_n \in B(x, 1/n) \cap A$. Then the sequence $\{x_n\}$ converges to $x$ with each $x_n \in A$. This is how we usually construe a limit point. We can generalize this idea to topological spaces.

DEFINITION 3.7. *A sequence $\{x_n\}$ of points in a topological space $(X, \mathcal{T})$ is said to* **converge to a point** $x \in X$, *if for any open set $U$ containing $x$, there is a positive integer $N = N(U)$ so that $x_n \in U$ whenever $n \geq N$.*

This definition includes the notion of convergence in a metric space. However, in a general topological space, convergence of a sequence can be very strange. For example, consider the following topology on a nonempty set $X$: Let $x_0 \in X$ be chosen once and for all. Define $\mathcal{T}_{IP} = \{\emptyset \text{ or } U \subset X \text{ with } x_0 \in U\}$. This gives a topology in $X$ called the **included point topology**. (Check for yourself that $\mathcal{T}_{IP}$ is a topology.) Suppose $\{x_n\}$ is the constant sequence of points, $x_n = x_0$ for all $n$. The sequence converges to $y \in X$, for *any* $y$: Any open set containing $y$, being nonempty, contains $x_0$. Thus *a constant sequence converges to every other point in the space* $(X, \mathcal{T}_{IP})$.

This example is extreme and it shows how wild an example a generalization can produce. Some further conditions keep such pathology in check. For example, to guarantee that a constant sequence converges only to the given point (and not other points as well), one needs at least one open set away from the point. The condition, $X$ is a $T_1$-space, introduced in the previous exercises, requires that singleton sets be closed. A constant sequence can converge only to itself because there is an open set separating other points from it. We next introduce another formulation of the $T_1$ condition, placing it in a family of such conditions.

DEFINITION 3.8. *A topological space $X$ is said to satisfy the $T_1$* **axiom** *(Trennungsaxiom) if given two points $x, y \in X$, there are open sets $U$, $V$ with $x \in U$, $y \notin U$ and $y \in V$, $x \notin V$. A topological space is said to satisfy the* **Hausdorff condition** *if given two points*

$x, y \in X$ *there are open sets* $U$, $V$ *with* $x \in U$, $y \in V$ *and* $U \cap V = \emptyset$. *The Hausdorff condition is also called the* $T_2$ **axiom**.

PROPOSITION 3.9. *A space* $X$ *satisfies the* $T_1$ *axiom if and only if a finite subset of points in* $X$ *is closed.*

*Proof:* Since a finite union of closed sets is closed, it suffices to check only a singleton subset. Suppose $x \in X$ and $X$ is $T_1$; we show that $\{x\}$ is closed. Let $y$ be in $X$, $y \neq x$. Then, by the $T_1$ axiom, there is an open set with $y \in U$, $x \notin U$. Denote this set by $U_y$. We have $U_y \subset X - \{x\}$. This can be done for each point $y \in X - \{x\}$ and we get

$$X - \{x\} \subset \bigcup_{y \in X - \{x\}} U_y \subset X - \{x\}.$$

Thus $X - \{x\}$ is a union of open sets, and $\{x\}$ is closed.

Conversely, suppose every singleton subset is closed in $X$. If $x$, $y \in X$ with $x \neq y$, then $x \in X - \{y\}$, $y \notin X - \{y\}$ and $X - \{y\}$ is open in $X$. Similarly, $y \in X - \{x\}$ and $x \notin X - \{x\}$, an open set in $X$. ♢

The $T_1$ axiom excludes some strange convergence behavior, but it is not enough to guarantee the uniqueness of limits. For example, if $(X, \mathcal{T}) = (\mathbb{R}, \mathcal{T}_{FC})$, the finite-complement topology on $\mathbb{R}$, then the $T_1$ axiom holds but the sequence of positive integers, $\{1, 2, 3, \ldots\}$ converges to every real number. The Hausdorff condition remedies this pathology.

THEOREM 3.10. *In a Hausdorff space, the limit of a sequence is unique.*

*Proof:* Suppose $\{x_n\}$ converges to $x$ and to $y$ with $x \neq y$. By the Hausdorff condition there are open sets $U$, $V$ with $x \in U$, $y \in V$ such that $U \cap V = \emptyset$. But the definition of convergence gives $N = N(U)$ and $M = M(V)$ with $x_n \in U$ for $n \geq N$ and $x_m \in V$ for $m \geq M$. Take $L = \max\{N, M\}$ and $x_\ell \in U \cap V$ for $\ell \geq L$. But this cannot be, so our assumption $x \neq y$ fails. ♢

An infinite set with the finite-complement topology is not Hausdorff.

A nice feature of the space ($\mathbb{R}$, usual) is its countable basis: thus open sets are expressible in a nice way. Another remarkable feature of $\mathbb{R}$ is the manner in which $\mathbb{Q}$ sits in $\mathbb{R}$. In particular, as we will see, cls $\mathbb{Q} = \mathbb{R}$. We relate these features in the general setting of topological spaces.

DEFINITION 3.11. *A subset* $A$ *of a topological space* $X$ *is* **dense** *if* cls $A = X$. *A topological space is* **separable** *(or Fréchet), if it has a countable dense subset.*

THEOREM 3.12. *A separable metric space is second countable.*

*Proof:* Suppose $A$ is a countable dense subset of $(X, d)$. Consider the collection of open balls

$$\{B(a, p/q) \mid a \in A, p/q > 0, p/q \in \mathbb{Q}\}.$$

If $U$ is an open set in $X$ and $x \in U$, then there is an $\epsilon > 0$ with $B(x, \epsilon) \subset U$. Since cls $A = X$, there is a point $a \in A \cap B(x, \epsilon/2)$. Consider $B(a, p/q)$ where $p/q$ is rational and $d(a, x) < p/q < \epsilon/2$. Then $x \in B(a, p/q) \subset B(x, \epsilon) \subset U$. Repeat this procedure for each $x \in U$ to show $U \subset \bigcup_a B(a, p/q) \subset U$ and this collection of open balls is a basis for the topology on $X$. The collection is countable since a countable union of countable sets is countable. $\diamondsuit$

The theorem applies to ($\mathbb{R}$,usual) and $\mathbb{Q} \subset \mathbb{R}$. Let $C^\infty([0,1],\mathbb{R})$ denote the set of all smooth functions $[0,1] \to \mathbb{R}$, that is, functions possessing continuous derivatives of every order. From real analysis we know that any smooth function on $[0,1]$ is bounded (a proof of this appears in Chapter 6) and so we can equip $C^\infty([0,1],\mathbb{R})$ with the metric $d(f,g) = \max_{t\in[0,1]}\{|f(t) - g(t)|\}$. The Stone-Weierstrass theorem ([?]) implies that the countable set of polynomials with rational coefficients is dense in $C^\infty([0,1],\mathbb{R})$. The proof follows by taking Taylor polynomials and approximating the coefficients by rationals. Thus $C^\infty([0,1],\mathbb{R})$ is second countable.

When we defined continuity of a function in the calculus, we first define what it means to be continuous at a point. This is a *local* notion that requires only information about the behavior of the function close to the point. To be continuous in the calculus, a function must be continuous at every point of its domain, and this is a *global* condition. The topological formulation of continuous is global, though it can be made local to a point. Many properties of spaces have a local variant that expresses dependence on a chosen point. For example, we give a local version of second countability.

DEFINITION 3.13. *A topological space is* **first countable** *if for each $x \in X$ there is a collection of open sets $\{U_i^x \mid i = 1,2,3,\ldots\}$ such that, for any $V$ open in $X$ with $x \in V$, there is one of these open sets $U_j^x$ with $x \in U_j^x \subset V$.*

The corresponding **global** condition is a countable basis for the entire space, that is, second countability.

The condition of first countability allows us to formulate the notion of limit point sequentially.

PROPOSITION. 3.14. *If $A \subset X$, a first countable space, then $x$ is in* cls *$A$ if and only if some sequence of points in $A$ converges to $x$.*

*Proof:* If $\{x_n\}$ is a sequence of points in $A$ converging to $x$, then any open set $V$ containing $x$ meets the sequence and we see either $x \in \text{int } A$ or $x \in \text{bdy}A$, so $x \in \text{cls } A$.

Conversely, if $x \in \text{cls } A$, consider the collection $\{U_i^x \mid 1 = 1,2,\ldots\}$ given by the condition of first countability. Then $A \cap U_1^x \cap U_2^x \cap \ldots \cap U_n^x \neq \emptyset$ for all $n$. Choose some $x_n \in A \cap U_1^x \cap \cdots \cap U_n^x$. The sequence $\{x_n\}$ converges to $x$: If $V$ is open in $X$ and $x \in V$, then there is $U_j^x$ with $x \in U_j^x \subset V$. But then $A \cap U_1^x \cap \ldots \cap U_m^x \subset U_j^x \subset V$ for all $m \geq j$, and so $x_m \in V$ for $m \geq j$. $\diamondsuit$

COROLLARY 3.15. *In a first countable space $X$, a subset $A \subset X$ is closed if and only if each point of $X$ for which $x = \lim_{n\to\infty} a_n$ for a sequence of points $a_n \in A$ satisfies $x \in A$.*

These ideas allow us to generalize the notion of sequential convergence as a criterion for continuity of functions.

In analysis it is useful to have various formulations of continuity, and so too in topology.

THEOREM 3.16. *Let $X, Y$ be topological spaces and $f\colon X \to Y$ a function. Then the following are equivalent:*

(1) *$f$ is continuous.*
(2) *If $K$ is closed in $Y$, then $f^{-1}(K)$ is closed in $X$.*

(3) *If $A \subset X$, then $f(\text{cls } A) \subset \text{cls } f(A)$.*

*Proof:* We first note that for any subset $S$ of $Y$,

$$\begin{aligned} f^{-1}(Y - S) &= \{x \in X \mid f(x) \in Y - S\} \\ &= \{x \in X \mid f(x) \notin S\} = \{x \in X | x \notin f^{-1}(S)\} \\ &= X - f^{-1}(S). \end{aligned}$$

(1) $\Longleftrightarrow$ (2): If $K$ is closed in $Y$, then $Y - K$ is open and, because $f$ is continuous, we have $f^{-1}(Y - K) = X - f^{-1}(K)$ is open in $X$. Thus $f^{-1}(K)$ is closed.

If $V$ is open in $Y$, then $f^{-1}(V) = X - f^{-1}(Y - V)$ and $Y - V$ is closed. So $f^{-1}(V)$ is open in $X$ and $f$ is continuous.

(2) $\Rightarrow$ (3): For $A \subset X$, cls $f(A)$ is closed in $Y$ and so $f^{-1}(\text{cls } (f(A)))$ is closed in $X$. It follows from $A \subset f^{-1}(f(A)) \subset f^{-1}(\text{cls } f(A))$, when $f^{-1}(\text{cls } f(A))$ is closed, that

$$\text{cls } A \subset f^{-1}(\text{cls } f(A))$$

and so $f(\text{cls } A) \subset \text{cls } f(A)$.

(3) $\Rightarrow$ (2): If $K$ is closed in $Y$, then $K = \text{cls } K$. Let $L = f^{-1}(K)$. We show cls $L \subset L$.

$$f(\text{cls } L) = f(\text{cls } f^{-1}(K)) \subset \text{cls } f(f^{-1}(K)) = \text{cls } K = K.$$

Taking inverse images, cls $L \subset f^{-1}(f(\text{cls } L)) \subset f^{-1}(K) = L$. ♢

Part (3) of the theorem says that continuous functions send limit points to limit points.

Corollary 3.17. *If $f : X \to Y$ is a continuous function, and $\{x_n\}$ a sequence in $X$ converging to $x$, then the sequence $\{f(x_n)\}$ converges to $f(x)$. Furthermore, if $X$ is first countable, then the converse holds.*

*Proof:* Suppose $\{x_n\}$ is a sequence of points in $X$ with $\lim_{n\to\infty} x_n = x \in X$. If $U \subset Y$ is open and $f(x) \in U$, then $x \in f^{-1}(U)$ which is open in $X$ since $f$ is continuous. Because $\lim_{n\to\infty} x_n = x$, there is an index $N_U$ with $x_m \in f^{-1}(U)$ for all $m \geq N_U$. This implies that $f(x_m) \in U$ for all $m \geq N_U$ and so $\lim_{n\to\infty} f(x_n) = f(x)$.

To prove the converse, we assume that $f\colon X \to Y$ is not continuous. Then there is a closed subset of $Y$, $K \subset Y$ for which $f^{-1}(K)$ is not closed in $X$. Since the empty set is closed, we know that $f^{-1}(K)$ and also $K$ are not empty. Furthermore, since $f^{-1}(K)$ is not closed, there is a point $x \in \text{cls } f^{-1}(K)$ for which $x \notin f^{-1}(K)$. Because $X$ is first countable, there is a sequence of points $\{x_n\}$ with $x_n \in f^{-1}(K)$ for all $n$ and $\lim_{n\to\infty} x_n = x$. Then $f(x_n) \in K$ for all $n$ and since $K$ is closed, $\lim_{n\to\infty} f(x_n) \in K$ if it exists. However, $\lim_{n\to\infty} f(x_n) \neq f(x)$ since $x \notin f^{-1}(K)$. ♢

The extent to which the problem of the invariance of dimension is disconcerting may be seen by the following example of a continuous function due to Guiseppe Peano (1858–1932).

Given a real number $r$ with $0 \leq r \leq 1$, we can represent it by its ternary expansion, that is,

$$r = 0.t_1 t_2 t_3 \cdots = \sum_{i=1}^{\infty} \frac{t_i}{3^i} \text{ where } t_i \in \{0, 1, 2\}.$$

Such a representation is unique except in the special cases:

$$r = 0.t_1t_2\cdots t_n222\cdots = 0.t_1t_2\cdots t_{n-1}(t_n+1)000\cdots, \text{ where } t_n \neq 2.$$

In an 1890 paper [?], Peano introduced a function defined on $[0,1]$ using the ternary expansion. Let $\sigma$ denote the permutation of $\{0,1,2\}$ which exchanges 0 and 2 and leaves 1 fixed. We will think of $\sigma$ as acting on the ternary digits of a number. The way in which this permutation acts can be understood by observing that when we write $r = 0.t_1t_2t_3\cdots$, in its ternary expansion, then

$$1 - r = 0.222\cdots - 0.t_1t_2t_3\cdots = 0.(\sigma t_1)(\sigma t_2)(\sigma t_3)\cdots.$$

Let $\sigma^t = \sigma\circ\sigma\circ\cdots\circ\sigma$ ($t$ times). We define $\mathrm{Pe}(r) = (0.a_1a_2a_3\cdots, 0.b_1b_2b_3\cdots)$ by

$$\begin{array}{ll} a_1 = t_1 & b_1 = \sigma^{t_1}t_2 \\ a_2 = \sigma^{t_2}t_3 & b_2 = \sigma^{t_1+t_3}t_4 \\ \vdots & \vdots \\ a_n = \sigma^{t_2+t_4+\cdots t_{2(n-1)}}t_{2n-1} & b_n = \sigma^{t_1+t_3+\cdots t_{2n-1}}t_{2n} \\ \vdots & \vdots \end{array}$$

From the definition of $\sigma$ and $\mathrm{Pe}$, the value of $\mathrm{Pe}(r)$ is the ternary expansions of a pair of real numbers $0 \leq x, y \leq 1$. The properties of the function $\mathrm{Pe}$ prompted Felix Hausdorff (1868–1942) to write [?] of it: "This is one of the most remarkable facts of set theory."

Theorem 3.18. *The function* $\mathrm{Pe}\colon [0,1] \longrightarrow [0,1]\times[0,1]$ *is well-defined, continuous, and onto.*

Because this function is onto a square in $\mathbb{R}^2$, it is called a **space-filling curve**. By changing the definition of the curve slightly, it can be made to be onto $[0,1]^{\times n} = [0,1]\times[0,1]\times\cdots\times[0,1]$ ($n$ times) for $n \geq 2$. We note that the function is not one-one and so fails to be a bijection. However, the fact that it is continuous indicates the subtlety of the problem of dimension.

*Proof:* We first put the Peano curve into a form that is convenient for our discussion. The definition given by Peano is recursive and so we use this feature to give another expression for the function.

$$\mathrm{Pe}(0.t_1t_2t_3\cdots) = (0.t_1, \sigma^{t_1}t_2) + (\sigma^{t_2}, \sigma^{t_1})\circ\frac{\mathrm{Pe}(0.t_3t_4t_5\cdots)}{3}.$$

Here, by $(\sigma^{t_2}, \sigma^{t_1})$, I mean the operation defined

$$\begin{aligned}(\sigma^{t_2}, \sigma^{t_1})&(0.a_1a_2a_3\cdots\ ,\ 0.b_1b_2b_3) \\ &= (0.(\sigma^{t_2}a_1)(\sigma^{t_2}a_2)(\sigma^{t_2}a_3)\cdots\ ,\ 0.(\sigma^{t_1}b_1)(\sigma^{t_1}b_2)(\sigma^{t_1}b_3)\cdots).\end{aligned}$$

We can now prove $\mathrm{PE}$ is well-defined. Using the recursive definition, we reduce the question of well-definedness to comparing the values $\mathrm{PE}(0.0222\cdots)$ and $\mathrm{PE}(0.1000\cdots)$ and the values $\mathrm{PE}(0.1222\cdots)$ and $\mathrm{PE}(0.2000\cdots)$. Applying the definition we find

$$\mathrm{PE}(0.0222\cdots) = (0.0222\cdots, 0.222\cdots) \text{ and } \mathrm{PE}(0.1000\cdots) = (0.1000\cdots, 0.222\cdots).$$

The ambiguity in ternary expansions implies $\mathrm{PE}(0.0222\cdots) = \mathrm{PE}(0.1000\cdots)$.

Similarly we have

$$\mathrm{PE}(0.1222\cdots) = (0.1222\cdots, 0.000\cdots) \text{ and } \mathrm{PE}(0.2000\cdots) = (0.2000\cdots, 0.000\cdots),$$

and so $\mathrm{PE}(0.1222\cdots) = \mathrm{PE}(0.2000\cdots)$.

We next prove that the mapping $\mathrm{PE}$ is onto. Suppose $(u, v) \in [0, 1] \times [0, 1]$. We write

$$(u, v) = (0.a_1a_2a_3\cdots, 0.b_1b_2b_3\cdots).$$

Let $t_1 = a_1$. Then $t_2 = \sigma^{t_1}b_1$. Since $\sigma \circ \sigma = \mathrm{id}$, we have $\sigma^{t_1}t_2 = \sigma^{t_1} \circ \sigma^{t_1}b_1 = b_1$. Next let $t_3 = \sigma^{t_2}a_2$. Continue in this manner to define

$$t_{2n-1} = \sigma^{t_2+t_4+\cdots t_{2(n-1)}}a_n, \qquad t_{2n} = \sigma^{t_1+t_3+\cdots+t_{2n-1}}b_n.$$

Then $\mathrm{PE}(0.t_1t_2t_3\cdots) = (0.a_1a_2a_3\cdots, 0.b_1b_2b_3\cdots) = (u, v)$ and $\mathrm{PE}$ is onto.

Finally, we prove that $\mathrm{PE}$ is continuous. We use the fact that $[0, 1]$ is a first countable space and show that for all $r \in [0, 1]$, whenever $\{r_n\}$ is a sequence of points in $[0, 1]$ with $\lim_{n\to\infty} r_n = r$, then $\lim_{n\to\infty} \mathrm{PE}(r_n) = \mathrm{PE}(r)$.

Suppose $r = 0.t_1t_2t_3\cdots$ has a unique ternary representation. For any $\epsilon > 0$, we can choose $N > 0$ with $\epsilon > 1/3^N > 0$. Then the value of $\mathrm{PE}(r)$ is determined up to the first $N$ ternary digits in each coordinate by the first $2N$ digits of the ternary expansion of $r$. For any sequence $\{r_n\}$ converging to $r$, there is an index $M = M(2N)$ with the property that for $m > M$, the first $2N$ ternary digits of $r_m$ agree with those of $r$. It follows that the first $N$ ternary digits of each coordinate of $\mathrm{PE}(r_m)$ agree with those of $\mathrm{PE}(r)$ and so $\lim_{n\to\infty} \mathrm{PE}(r_n) = \mathrm{PE}(r)$.

In the case that $r$ has two ternary representations,

$$r = 0.t_1t_2t_3\cdots t_N000\cdots = 0.t_1t_2t_3\cdots(t_N - 1)222\cdots,$$

with $t_N \neq 0$, we can apply the familiar trick of the calculus of considering convergence from above or below the value $r$. Suppose that $\{r_n\}$ is a sequence in $[0, 1]$ with $\lim_{n\to\infty} r_n = r$ and $r \leq r_n$ for all $n$. Then for some index $M$, when $m > M$ we have $r_m = 0.t_1t_2t_3\cdots t_Nt'_{N+1}t'_{N+2}\cdots$. We can now argue as above that $\lim_{n\to\infty} \mathrm{PE}(r_n) = \mathrm{PE}(r)$. On the other side, for a sequence $\{s_n\}$ with $\lim_{n\to\infty} s_n = r$ and $s_n \leq r$ for all $n$, we compare $s_n$ with $r = 0.t_1t_2t_3\cdots(t_N - 1)222\cdots$. Once again, we eventually have that $s_m = 0.t_1t_2t_3\cdots(t_N - 1)t''_{N+1}t''_{N+2}\cdots$. Convergence of the series $\{s_n\}$ implies that more of ternary expansion agrees with $r$ as $n$ grows larger, and so $\lim_{n\to\infty} \mathrm{PE}(s_n) = \mathrm{PE}(r)$.

Since convergence from each side implies general convergence, we have proved that PE is continuous. ♢

To get a useful picture of the Peano mapping consider the recursive expression.

$$\text{PE}(0.t_1t_2t_3\cdots) = (0.t_1, \sigma^{t_1}t_2) + (\sigma^{t_2}, \sigma^{t_1}) \circ \frac{\text{PE}(0.t_3t_4t_5\cdots)}{3}.$$

When $r$ is in the first ninth of the unit interval, we can write $r = 0.00t_3t_4\cdots$ and so $\text{PE}(r) = \text{PE}(0.t_3t_4t_5\cdots)/3$. Since $0.t_3t_4\cdots$ varies over the entire line segment $[0,1]$, there is a copy of the image of the interval, shrunk to fit into the lower left corner of the $3\times 3$ subdivided square, ending at the point $(1/3, 1/3)$. The second ninth of $[0,1]$ consists of $r$ with $r = 0.01t_3t_4\cdots$ and so we find $\text{PE}(r) = (0, 0.1) + (\sigma, \text{id}) \circ (\text{PE}(0.t_3t_4t_5\cdots)/3)$. Thus the copy of the image of the interval is shrunk by a factor of 3, flipped by the mapping $(x, y) \mapsto (1-x, y)$, a reflection across the vertical midline of the square, and then translated up by adding $(0, 0.1)$. This places the image of the origin at the point $(0.1, 0.1)$ and ties the end of the image of the first ninth to the beginning of the image of the second ninth. The well-definedness of PE is at work here.

| | | |
|---|---|---|
| 02 | 10 | 22 |
| 01 | 11 | 21 |
| 00 | 12 | 20 |

If we put the first two digits of the ternary expansion of $r$ into the appropriate subsquare, we get the pattern above and the image of the interval, shrunk to fit each subsquare, fills each subsquare oriented by the action of $\sigma$ where

$$(\sigma, \text{id}) \leftrightarrow (1-x, y); (\text{id}, \sigma) \leftrightarrow (x, 1-y);\text{ and } (\sigma, \sigma) \leftrightarrow (1-x, 1-y).$$

For example, the center subsquare, labeled 11, has a copy of the shrunken image of the interval upside down.

There are many approaches to space-filling curves. We have followed [?] in this exposition. Later, we will see that the failure of the Peano curve to be both onto and one-one is a feature of the topology of the unit interval and the unit square. For more discussion of the remarkable phenomenon of space-filling curves, see the book [?].

**Exercises**

1. Some statements about the closure operation: (1) Suppose that $A$ is dense in $X$ and $U$ is open in $X$. Show that $U \subset \text{cls }(A \cap U)$. (2) If $A$, $B$ and $A_\alpha$ are subsets of a topological space $X$, show that $\text{cls }(A \cup B) = \text{cls }(A) \cup \text{cls }(B)$. However, show that $\bigcup_\alpha \text{cls }(A_\alpha) \subset \text{cls }(\bigcup_\alpha A_\alpha)$. Give an example where the inclusion is proper. (3) Show that $\text{bdy}(A) = \text{cls }(A) \cap \text{cls }(X - A)$.

2. A subset $A \subset X$, a topological space, is called **perfect** if $A = A'$, that is, $A$ is identical with its derived set. Show that the Cantor set obtained by removing middle thirds from $[0, 1]$ is a perfect subset of $\mathbb{R}$.

3. A topological space $X$ is called a **metrizable space** if the topology on $X$ can be induced by a metric space structure on $X$. Not every topology on a set comes about in this fashion. Show that a metric space is always Hausdorff and first countable.

4. Suppose that $X$ is an uncountable set and that $x_0$ is a given point in $X$. Let $\mathcal{T}_F$ denote the Fort topology on $X$, $\{U \mid X - U \text{ is finite or } x_0 \notin U\}$.
   i) Show that $(X, \mathcal{T}_F)$ is a Hausdorff space.
   ii) Show that $(X, \mathcal{T}_F)$ is not first countable (and hence not metrizable).

5. Suppose that $(X, d)$ is a metric space and $A \subset X$. Define the *distance from* $A$ *to a point* $x$, $d(x, A)$ to be the infimum of the set of real numbers $\{d(x, a) \mid a \in A\}$.
   i) Show that $d(-, A) \colon X \to \mathbb{R}$ is a continuous function.
   ii) Show that a point $x \in X$ is in the closure of $A$ if and only if $d(x, A) = 0$.
   iii) What is the preimage of the closed subset $\{0\}$ of $\mathbb{R}$ under the mapping $d(-, A)$?

6. Prove that the following are topological properties: (1) $X$ is a separable space. (2) $X$ satisfies the Hausdorff condition. (3) $X$ has the discrete topology.