8^{bis}

Nontrivial fundamental groups

> "One of the advantages of the category of nilpotent spaces over that of simply-connected spaces is that it is closed under certain constructions."
>
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The category of simply-connected spaces is blessed with certain features that make homotopy theory tractable. In the first place, there is the Whitehead Theorem (Theorem 4.5) that tells us when a mapping of spaces of the homotopy type of CW-complexes is a homotopy equivalence—the necessary condition that the mapping induces an isomorphism of integral homology groups is also sufficient. Secondly, the Postnikov tower of a simply-connected space is a tower of principal fibrations pulled back via the k-invariants of the space (Theorem 8^{bis}.37). This makes cohomological obstruction theory accessible, if not computable ([Brown, E57], [Schön90], [Sergeraert94]). Furthermore, the system of local coefficients that arises in the description of the E_2-term of the Leray-Serre spectral sequence is simple when the base space of a fibration is simply-connected, and the cohomology Eilenberg-Moore spectral sequence converges strongly for a fibration pulled back from such a fibration.

A defect of the category of simply-connected spaces is the fact that certain constructions do not stay in the category. The dishearteningly simple example is the based loop space functor—if (X, x_0) is simply-connected, $\Omega(X, x_0)$ need not be. Furthermore, the graded group-valued functor, the homotopy groups of a space, does not always distinguish distinct homotopy types of spaces that are not simply-connected. A classic example is the pair of spaces $X_1 = \mathbb{R}P^{2m} \times S^{2n}$ and $X_2 = S^{2m} \times \mathbb{R}P^{2n}$; the homotopy groups in each degree k are abstractly isomorphic, $\pi_k(X_1) \cong \pi_k(X_2)$. If we had principal Postnikov towers, we could use the abstract isomorphisms to try to build a weak homotopy equivalence. However, the cohomology rings over $\mathbb{F}_2$ "know" that X_1 is not homotopy equivalent to X_2.

In this chapter we introduce the larger category of nilpotent spaces. These spaces enjoy some of the best homotopy-theoretic properties of simply-connected spaces, like a Whitehead theorem ([Dror71]) and reasonable Postnikov towers.

Furthermore, this category is closed under many constructions such as the formation of function spaces. Group-theoretic functors, like localization and completion, have topological extensions in this category. A subtler result related to nilpotent spaces is due to [Dwyer74] who showed that the Eilenberg-Moore spectral sequence for the fibre of a fibration converges strongly if and only if the action of $\pi_1(B)$ on $H_i(F;A)$ is nilpotent for all $i \geq 0$ (Theorem 8^{bis}.29).

In $\S 8^{\text{bis}}$.1, we discuss the various actions of a nontrivial fundamental group. The action of a group on a module leads to a right exact functor, the coinvariants of the group action, whose left-derived functors are the homology groups of a group. This theory is developed briefly in $\S 8^{\text{bis}}$.2 with an eye to its application to nilpotent spaces. In particular, we construct the Lyndon-Hochschild-Serre spectral sequence associated to a group extension, and the Cartan-Leray spectral sequence associated to a group acting freely and properly on a space.

With these tools in place we study the category of nilpotent groups and spaces in $\S 8^{\text{bis}}$.3. We first prove the generalized Whitehead Theorem of [Dror71]. We then discuss the Postnikov tower of a nilpotent space. This tower characterizes such spaces and provides a tool for making new spaces such as the localization of a space à la [Sullivan71]. Cosimplicial methods offer a functorial route to localization and we give a short survey of the foundational work of [Bousfield-Kan72]. We also prove Dwyer's convergence theorem for the Eilenberg-Moore spectral sequence. The (co)simplicial constructions described here have proved to be fundamental in homotopy theory. We end with a theorem of [Dror73] that, for connected spaces, any homotopy type can be approximated up to homology equivalence by a tower of nilpotent spaces.

§8^{bis}.1 Actions of the fundamental group

We begin with a small digression. Let (G, e, μ) denote a topological group with identity element e and write $g \cdot h$ for $\mu(g, h)$. It is an elementary fact that $\pi_0(G)$ is a group with multiplication induced by μ. There is an action of $\pi_0(G)$ on $\pi_n(G, e)$ defined as follows: If $g \in [g] \in \pi_0(G)$ is a point in a path component of G and $\alpha\colon (S^n, e_1) \to (G, e)$ represents a class $[\alpha]$ in $\pi_n(G, e)$, consider

$$g \cdot \alpha \cdot g^{-1}\colon (S^n, e_1) \longrightarrow (G, e), \quad \text{defined by} \;\; g \cdot \alpha \cdot g^{-1}(x) = g \cdot \alpha(x) \cdot g^{-1}.$$

When we vary the choice of g in $[g]$ or the choice of representative for $[\alpha]$, we get homotopic maps and so this recipe determines a pairing

$$\pi_0(G) \times \pi_n(G, e) \xrightarrow{\nu_n} \pi_n(G, e), \quad ([g], [\alpha]) \mapsto [g \cdot \alpha \cdot g^{-1}].$$

Since the multiplication on $\pi_0(G)$ is determined by $[g] \cdot [h] = [g \cdot h]$, we have $\nu_n([g] \cdot [h], [\alpha]) = \nu_n([g], [h] \cdot [\alpha])$. Furthermore, the addition on $\pi_n(G, e)$ agrees with the operation induced by μ and so we have $\nu_n([g], [\alpha] + [\beta]) = \nu_n([g], [\alpha]) + \nu_n([g], [\beta])$. Thus $\pi_n(G, e)$ is a module over the group $\pi_0(G)$.

When (X,x_0) is a pointed space of the homotopy type of a countable CW-complex, [Milnor56] showed how to replace the based loop space $\Omega(X,x_0)$ with a topological group. In this case we write $\bar{\nu}_n^X = \nu_{n-1}\colon \pi_0(\Omega(X,x_0)) \times \pi_{n-1}(\Omega(X,x_0)) \to \pi_{n-1}(\Omega(X,x_0))$ and we have proved the following theorem.

Theorem 8^{bis}.1. *Given a connected, pointed space (X,x_0), for each $n \geq 1$, $\pi_n(X,x_0)$ is a module over $\pi_1(X,x_0)$ via the pairing*

$$\bar{\nu}_n\colon \pi_1(X,x_0) \times \pi_n(X,x_0) \longrightarrow \pi_n(X,x_0).$$

Furthermore, when $n=1$, $\pi_1(X,x_0)$ acts on itself by conjugation, that is, $\bar{\nu}_1([\omega],[\lambda]) = [\omega]\cdot[\lambda]\cdot[\omega]^{-1} = [\omega * \lambda * \omega^{-1}]$ where $*$ denotes composition of paths.

In our discussion of the failure of the homotopy groups to distinguish the pair of spaces $\mathbb{R}P^{2m} \times S^{2n}$ and $S^{2m} \times \mathbb{R}P^{2n}$, we only used the observation that, for all k, $\pi_k(\mathbb{R}P^{2m} \times S^{2n}) \cong \pi_k(\mathbb{R}P^{2n} \times S^{2m})$, *as groups*. They differ as π_1-modules. It is a consequence of the interpretation of the action of the fundamental group as deck transformations that $\pi_1 = \mathbb{Z}/2\mathbb{Z}$ acts nontrivially on the $\mathbb{Z}$ factor in $\pi_{2m}(\mathbb{R}P^{2m} \times S^{2n})$ and $\pi_{2n}(\mathbb{R}P^{2n} \times S^{2m})$ coming from the projective space. Since these factors of $\mathbb{Z}$ as nontrivial $\mathbb{Z}/2\mathbb{Z}$-modules occur in different dimensions, the spaces could not be homotopy equivalent, and the homotopy groups, considered as graded π_1-modules, distinguish the spaces as different.

The fundamental group acts on other groups when we have a fibration $F \hookrightarrow E \xrightarrow{p} B$ with connected fibre F. Consider the long exact sequence of homotopy groups:

$$\cdots \xrightarrow{\partial} \pi_n(F,e) \xrightarrow{i_*} \pi_n(E,e) \xrightarrow{p_*} \pi_n(B,p(e)) \xrightarrow{\partial} \pi_{n-1}(F,e) \xrightarrow{i_*} \cdots .$$

We can induce an action of $\pi = \pi_1(E,e)$ on $\pi_n(B,p(e))$ via the composite

$$\nu_n^{E,B}\colon \pi \times \pi_n(B,p(e)) \xrightarrow{p_* \times 1} \pi_1(B,p(e)) \times \pi_n(B,p(e)) \xrightarrow{\bar{\nu}_n^B} \pi_n(B,p(e)).$$

Thus we can view $\pi_n(B,p(e))$ as a $\pi_1(E,e)$-module and the mapping p_* as a module homomorphism. In fact, more is true.

Proposition 8^{bis}.2. *When $F \hookrightarrow E \xrightarrow{p} B$ is a fibration of connected spaces, the long exact sequence on homotopy is a long exact sequence of $\pi_1(E,e)$-modules and module homomorphisms.*

PROOF: We first construct the action of $\pi = \pi_1(E,e)$ on $\pi_n(F,e)$ here taken to be $\pi_{n-1}(\Omega F, c_e)$. Let $\alpha\colon (S^{n-1},\vec{e}_1) \to (\Omega F, c_e)$ represent $[\alpha] \in$

$\pi_{n-1}(\Omega F, c_e)$ and $\omega \in [\omega] \in \pi_1(E, e)$. The action of π on $\pi_{n-1}(\Omega E, c_e)$ associates to $[\omega]$ and $[i \circ \alpha]$ the class $[\beta] = \nu^E_{n-1}([\omega], [i \circ \alpha])$, where $\beta(\vec{u}) = \omega * i \circ \alpha(\vec{u}) * \omega^{-1}$. The mapping $\Omega p\colon \Omega E \to \Omega B$ takes β to

$$p \circ \beta(\ ) = (p \circ \omega) * (p \circ i \circ \alpha(\ )) * (p \circ \omega)^{-1} = (p \circ \omega) * c_{p(e)} * (p \circ \omega)^{-1} \simeq c_{p(e)}.$$

Let $h\colon I \times I \to B$ denote a pointed homotopy between $p \circ \beta$ and $c_{p(e)}$. Construct the homotopy $H\colon S^{n-1} \times I \to \Omega B$ given by $H(\vec{u}, t)(r) = h(r, t)$. Thus $H(\vec{u}, 0)(r) = (p \circ \omega) * c_{p(e)} * (p \circ \omega)^{-1}(r)$ and $H(\vec{u}, 1)(r) = p(e)$. Consider the homotopy lifting problem posed by the diagram:

$$\begin{tikzcd}
S^{n-1} \times \{0\} \arrow[r, "\beta"] \arrow[d, hook] & (\Omega E, c_e) \arrow[d, "\Omega p"] \\
S^{n-1} \times I \arrow[r, "H"'] \arrow[ur, dashed, "\hat{H}"] & (\Omega B, c_{p(e)}).
\end{tikzcd}$$

Since Ωp is a fibration, there is a lifting $\hat{H}\colon S^{n-1} \times I \to (\Omega E, c_e)$ with $\hat{H}(\vec{u}, 0)(r) = \beta(\vec{u})(r)$. Because $p \circ \hat{H} = H$, the mapping $\beta'(\vec{u})(r) = \hat{H}(\vec{u}, 1)(r)$ determines a loop in F. We define the action of π on $\pi_{n-1}(\Omega F, c_e)$ to be the rule that associates $[\beta']$ to $[\omega] \in \pi$ and $[\alpha] \in \pi_{n-1}(\Omega F, c_e)$. By the properties of fibrations this is well-defined and gives a module action. We denote this action by $\nu^{E,F}_n\colon \pi_1(E, e) \times \pi_{n-1}(\Omega F, c_e) \longrightarrow \pi_{n-1}(\Omega F, c_e)$.

Notice that $i \circ \beta' \simeq \beta$. Because the class $[\beta]$ represents $\nu^E_{n-1}([\omega], [i \circ \alpha])$, we have that $i_*(\nu^{E,F}_{n-1}([\omega], [\alpha]) = \nu^E_{n-1}([\omega], i_*([\alpha]))$ and the homomorphism $i_*\colon \pi_{n-1}(\Omega F, c_e) \to \pi_{n-1}(\Omega E, c_e)$ is a homomorphism of π-modules. Furthermore, if $[\omega] = i_*([\omega'])$ for $[\omega'] \in \pi_1(F, e)$, then $i_*(\nu^F_{n-1}([\omega'], [\alpha])) = \nu^E_{n-1}(i_*([\omega']), i_*([\alpha]))$.

Finally, we consider the transgression $\partial\colon \pi_n(B, p(e)) \to \pi_{n-1}(F, e)$. It is best here to substitute a geometric mapping for ∂. Consider the pullback diagram:

$$\begin{tikzcd}
\Omega B \arrow[r, "j"] & E_{\mathrm{ev}_1, p} \arrow[r] \arrow[d, "q"'] & PB \arrow[d, "\mathrm{ev}_1"] \\
 & E \arrow[r, "p"'] & B.
\end{tikzcd}$$

Here $\mathrm{ev}_1\colon PB \to B$ is the path-loop fibration. Because PB is contractible, $E_{\mathrm{ev}_1, p}$ has the homotopy type of F and $j_* = \partial$. The space $E_{\mathrm{ev}_1, p}$ is the subspace of $E \times PB$ given by $\{(y, \lambda) \mid y \in E, \lambda\colon I \to B$, such that $\lambda(0) = p(e), \lambda(1) = p(y)\}$. The mapping $j\colon \Omega B \to E_{\mathrm{ev}_1, p}$ is given by $j(\gamma) = (e, \gamma)$. The action of a loop $\omega \in [\omega] \in \pi$ may be expressed at this level as $(e, \gamma) \mapsto (e, (p \circ \omega) * \gamma * (p \circ \omega)^{-1})$. In the definition of the action of π on $\pi_n(\Omega B, c_{p(e)})$, the mapping j_* takes the class $\nu^B_n([\omega], [\alpha])$ to $\bar{\nu}^{F,E}_{n-1}([\omega], j_*([\alpha]))$ after the identification of the fibre with $E_{\mathrm{ev}_1, p}$. Thus $j_* = \partial$ is a π-homomorphism.

After all the relevant identifications are made, we have shown that the long exact sequence

$$\cdots \to \pi_n(F,e) \xrightarrow{i_*} \pi_n(E,e) \xrightarrow{p_*} \pi_n(B,p(e)) \xrightarrow{\partial} \pi_{n-1}(F,e) \to \cdots$$

is an exact sequence of π-modules and π-module homomorphisms. □

In certain cases the π_1-action is trivial, even for π_1 a nontrivial group. For example, if X is an H-space, we have the following result of [Serre51].

Corollary 8^{bis}.3. *If (X,μ,e) is an H-space, then the action $\bar{\nu}_n^X$ of $\pi_1(X,e)$ on $\pi_n(X,e)$ is trivial for all n.*

PROOF: Let $\omega \in [\omega] \in \pi_1(X,e)$. If $\alpha\colon (S^{n-1},e_1) \to (\Omega X, c_e)$ represents a class in $\pi_n(X,e)$, then consider the homotopy $H\colon S^{n-1}\times I \to (\Omega X, c_e)$ given by $H(\vec{u},t)(r) = \mu((c_e * \alpha(\vec{u}) * c_e)(r), h(r,t))$ where $h\colon I\times I \to X$ is a pointed homotopy between c_e and $\omega * c_e * \omega^{-1}$. From the definition $H(\vec{u},0)(r) = \mu((c_e*\alpha(\vec{u})*c_e)(r),e) \simeq \alpha(\vec{u})(r)$ and $H(\vec{u},1)(r) = (\omega*\alpha(\vec{u})*\omega^{-1})(r)$. It follows that $\nu_{n-1}^X([\omega],[\alpha]) = [\alpha]$. □

This result extends the fact that the fundamental group of an H-space is abelian.

Another rich source of actions of the fundamental group is the notion of bundles of groups (§5.3) over a space. For example, when $F \hookrightarrow E \to B$ is a fibration, then $\pi_1(B,b_0)$ acts on the homology of F_{b_0}, the fibre over b_0. In Chapters 5 and 6 the applications of the Leray-Serre spectral sequence involved simple systems of local coefficients, that is, where $\pi_1(B,b_0)$ acts trivially on $H_i(F_{b_0};R)$. When the bundle of groups is not simple, then the E^2-term of the spectral sequence need not be a product, even for field coefficients. In the next proposition we give the first case of such a difference. The functor $\Gamma_\pi^2(\ )$ on π-modules defined in the proposition is the first of a family described fully in Definition 8^{bis}.17.

Proposition 8^{bis}.4. *Suppose $\mathcal{G}$ is a bundle of abelian groups over a pointed, path-connected space (X,x_0) and $G_0 = G_{x_0}$. Then $H_0(X;\mathcal{G}) \cong G_0/\Gamma_\pi^2 G_0$, where $\Gamma_\pi^2 G_0$ is the subgroup of G_0 generated by all elements of the form $[\alpha]\cdot g - g$ with $[\alpha] \in \pi = \pi_1(X,x_0)$ and $g \in G_0$.*

PROOF: Consider the mapping $\phi\colon G_0 \to H_0(X;\mathcal{G})$ given by $g \mapsto g\otimes x_0$. Suppose that $\alpha \in [\alpha] \in \pi_1(X,x_0)$. Then we associate to $g \in G_0$ and $\alpha\colon (\Delta^1,\partial\Delta^1) \to (X,x_0)$ the element $g\otimes\alpha$ in $C_1(X;\mathcal{G})$. The boundary of $g\otimes\alpha$ is given by the formula in §5.3:

$$\partial_h(g\otimes\alpha) = h[\alpha^{-1}](g)\otimes\alpha(1) - g\otimes\alpha(0) = (h[\alpha^{-1}](g) - g)\otimes x_0.$$

Since α and g were arbitrary, we see that ϕ takes $\Gamma^2_\pi G_0$ to 0 and so induces a homomorphism $\bar{\phi}\colon G_0/\Gamma^2_\pi G_0 \to H_0(X;\mathcal{G})$.

To show that $\bar{\phi}$ is an isomorphism, we describe its inverse. Suppose $u \in H_0(X;\mathcal{G})$. We can write

$$u = \sum\nolimits_{i=1}^n g_i \otimes x_i + B_0(X;\mathcal{G}).$$

Let $\lambda_i\colon [0,1] \to X$ denote a path in X joining x_0 to x_i. Then we have the isomorphism $h[\lambda_i]\colon G_{x_i} \to G_0$ for each i. We define $\psi\colon H_0(X;\mathcal{G}) \to G_0/\Gamma^2_\pi G_0$ by

$$\psi\left(\sum\nolimits_{i=1}^n g_i \otimes x_i + B_0(X;\mathcal{G})\right) = \sum\nolimits_{i=1}^n h[\lambda_i](g_i).$$

To see that ψ is well-defined, we notice that ψ takes boundaries to zero—let $h[\alpha^{-1}](g) \otimes \alpha(1) - g \otimes \alpha(0)$ denote a generator of $B_0(X;\mathcal{G})$. The homomorphism ψ applied to such an element gives $h[\beta_1](h[\alpha^{-1}](g)) - h[\beta_0](g)$ where β_i is a path in X starting at x_0 and ending at $\alpha(i)$. There is a loop based at x_0 given by $\beta_0 * \alpha * \beta_1^{-1}$ and

$$\begin{aligned} &h[\beta_0 * \alpha * \beta_1^{-1}](h[\beta_1](h[\alpha^{-1}](g)) - h[\beta_0](g)) \\ &\quad = h[\beta_0](g) - h[\beta_0 * \alpha * \beta_1^{-1}](h[\beta_0](g)), \end{aligned}$$

which is an element of $\Gamma^2_\pi G_0$. Since $\Gamma^2_\pi G_0$ is closed under the action of $\pi_1(X, x_0)$, ψ takes $B_0(X;\mathcal{G})$ to zero.

We must account for the choices made in the construction of ψ. If λ_i and μ_i are paths joining x_0 to x_i, then we compare $h[\lambda_i](g_i)$ and $h[\mu_i](g_i)$, the images of $g_i \otimes x_i$ with respect to the different paths. The difference between these values can be rewritten

$$\begin{aligned} h[\lambda_i](g_i) - h[\mu_i](g_i) &= h[\mu_i * \lambda_i^{-1}]h[\lambda_i * \mu_i^{-1}](h[\lambda_i](g_i) - h[\mu_i](g_i)) \\ &= h[\mu_i * \lambda_i^{-1}](h[\lambda_i * \mu_i^{-1}](h[\lambda_i](g_i)) - h[\lambda_i](g_i)), \end{aligned}$$

which is an element in $\Gamma^2_\pi G_0$. Thus ψ is well-defined and the inverse of $\bar{\phi}$. $\square$

By Theorem 5.1, it follows for an arbitrary fibration with path-connected base and connected fibre, $F \hookrightarrow E \to B$, that the associated Leray-Serre spectral sequence has $E^2_{0,*} \cong H_*(F;k)/\Gamma^2_\pi H_*(F;k)$ as the leftmost column of the E^2-term. Though this seems to put us in murkier waters, we can still see our way to deep results in homotopy theory by studying abstractly the action of groups on abelian groups.

§8^{bis}.2 Homology of groups

If π denotes a group and M a module over a ring R, then we say that π **acts on** M, or M is a π**-module**, if there is a homomorphism $\rho\colon \pi \to \mathrm{Aut}_R(M)$, where $\mathrm{Aut}_R(M)$ is the group of R-linear isomorphisms of M to itself. More generally, π acts on a group G if there is a homomorphism $\rho\colon \pi \to \mathrm{Aut}(G)$. To study modules over a group π we introduce the homology of groups, a homological functor analogous to Tor. The homology groups of a group π with coefficients in a π-module M satisfy the axioms for the left-derived functors ([Cartan-Eilenberg56]) of the functor that associates to a π-module M its **coinvariants**,

$$(\ )_\pi\colon M \mapsto M_\pi = M/\Gamma^2_\pi M.$$

Here $\Gamma^2_\pi M$ is the submodule of M generated by elements of the form $am - m$ where $a \in \pi$ and $m \in M$. (More generally, if G is a nonabelian group acted on by π, let $\Gamma^2_\pi G$ be the normal subgroup generated by elements of the form $(ag)g^{-1}$.) Since $b(am-m) = (bab^{-1})bm - bm$, $\Gamma^2_\pi M$ is a π-module. Thus a π-equivariant module homomorphism induces a homomorphism on coinvariants. The induced action of π on M_π is trivial. In fact, M_π may be characterized as the *largest* quotient of M on which π acts trivially. By 'largest' quotient we mean that, if M/M' is another quotient of M by a π-submodule M' and π acts trivially on M/M', then $\Gamma^2_\pi M \subset M'$ and there is an epimorphism $M/\Gamma^2_\pi M \twoheadrightarrow M/M'$.

To see that the functor $(\ )_\pi$ is right exact, we give another expression for M_π when M is a π-module.

Lemma 8^{bis}.5. *Let $\mathbb{Z}$ denote the ring of integers, taken as a trivial right π-module. If M is a π-module, then $M_\pi \cong \mathbb{Z} \otimes_{\mathbb{Z}\pi} M$, where $\mathbb{Z}\pi$ denotes the integral group ring of π.*

PROOF: Recall that $\mathbb{Z} \otimes_{\mathbb{Z}\pi} M$ is defined as the cokernel in the sequence

$$\mathbb{Z} \otimes \mathbb{Z}\pi \otimes M \xrightarrow{\psi\otimes 1 - 1\otimes\phi} \mathbb{Z} \otimes M \to \mathbb{Z} \otimes_{\mathbb{Z}\pi} M \to 0,$$

where $\phi\colon \mathbb{Z}\otimes\mathbb{Z}\pi \to \mathbb{Z}$ is the trivial right action of π on $\mathbb{Z}$, and $\psi\colon \mathbb{Z}\pi\otimes M \to M$ is the left action of π on M. In this case, $\otimes = \otimes_{\mathbb{Z}}$ and so the sequence becomes

$$\mathbb{Z}\pi \otimes M \xrightarrow{\eta} M \to \mathbb{Z} \otimes_{\mathbb{Z}\pi} M \to 0,$$

where $\eta(a \otimes m) = m - am$. Thus $\mathbb{Z} \otimes_{\mathbb{Z}\pi} M = M/\operatorname{im}\eta = M/\Gamma^2_\pi M$. □

The lemma implies that $(\ )_\pi$ is a right exact functor since $\mathbb{Z} \otimes_{\mathbb{Z}\pi} (\ )$ is right exact.

Definition 8^{bis}.6. *The* **homology of a group π with coefficients in a (left) π-module** M *is defined by*

$$H_i(\pi, M) = \mathrm{Tor}_i^{\mathbb{Z}\pi}(\mathbb{Z}, M).$$

We write $H_i(\pi)$ for $H_i(\pi, \mathbb{Z})$ when $\mathbb{Z}$ is the trivial left π-module.

To compute group homology we introduce a convenient functorial resolution. Let $\mathtt{B}_n$ denote the free abelian group on $(n+1)$-tuples of elements of π. Then $\mathtt{B}_n$ is a right π-module with the π-diagonal action

$$(x_0, \dots , x_n)a = (x_0 a, \dots , x_n a).$$

As a $\mathbb{Z}\pi$-module, $\mathtt{B}_n$ is free on those $(n+1)$-tuples with last entry 1. Consider the complex

$$\cdots \longrightarrow \mathtt{B}_n \xrightarrow{\partial} \mathtt{B}_{n-1} \xrightarrow{\partial} \cdots \xrightarrow{\partial} \mathtt{B}_2 \xrightarrow{\partial} \mathtt{B}_1 \xrightarrow{\partial} \mathbb{Z}\pi \xrightarrow{\varepsilon} \mathbb{Z} \to 0$$

where $\partial = \sum_i (-1)^i \bar{d}_i$ and $\bar{d}_i(x_0, \dots , x_n) = (x_0, \dots , \widehat{x_i}, \dots , x_n)$; ε is the usual group ring augmentation given by $\varepsilon\left(\sum_i n_i a_i\right) = \sum_i n_i$. This complex is exact since there is the contracting homotopy defined by $s(x_0, \dots , x_n) = (1, x_0, \dots , x_n)$ and $s(1) = 1$. To make the free $\mathbb{Z}\pi$-module structure evident, we introduce the **bar notation** for generators over $\mathbb{Z}\pi$: For $n = 0$, $[\ ] = 1$, and for $n > 0$,

$$[x_1 \mid \cdots \mid x_n] = (x_1 x_2 \cdots x_n,\ x_2 x_3 \cdots x_n,\ \dots ,\ x_{n-1} x_n,\ x_n,\ 1) \in \mathtt{B}_n.$$

The differential ∂ can be rewritten as $\partial = \sum_i (-1)^i d_i$ where

$$d_i([x_1 \mid \cdots \mid x_n]) = \begin{cases} [x_2 \mid \cdots \mid x_n], & \text{for } i = 0, \\ [x_1 \mid \cdots \mid x_{i-1} \mid x_i x_{i+1} \mid \cdots \mid x_n], & \text{for } 0 < i < n, \\ [x_1 \mid \cdots \mid x_{n-1}] x_n, & \text{for } i = n. \end{cases}$$

This is the familiar **bar construction** on π that gives a functorial free right $\mathbb{Z}\pi$-module resolution of the trivial module $\mathbb{Z}$.

Although it is large, the bar construction can still be used to prove structure results.

Proposition 8^{bis}.7. *If π is a finite group of order $|\pi|$ and M is a π-module, then every element of $H_i(\pi, M)$ for $i > 0$ has order a divisor of $|\pi|$.*

PROOF: Consider the π-module homomorphism , $s\colon \mathtt{B}_n \to \mathtt{B}_{n+1}$, given on generators by $s([x_1 \mid \cdots \mid x_n]) = \sum y \in \pi [y \mid x_1 \mid \cdots \mid x_n]$. We show that $\partial \circ s + s \circ \partial = |\pi|\,\mathrm{id}$:

$$\begin{aligned} \partial \circ s([x_1 \mid \cdots \mid x_n]) &= \partial \left(\sum\nolimits_{y\in\pi} [y \mid x_1 \mid \cdots \mid x_n] \right) \\ &= \sum\nolimits_{y\in\pi} \sum\nolimits_i (-1)^i d_i([y \mid x_1 \mid \cdots \mid x_n]) \\ &= |\pi|[x_1 \mid \cdots \mid x_n] - \sum\nolimits_{y\in\pi} \Big\{ [y x_1 \mid \cdots \mid x_n] - [y \mid x_1 x_2 \mid x_3 \mid \cdots \mid x_n] \\ &\qquad + \cdots + (-1)^{i+1} [y \mid x_1 \mid \cdots \mid x_i x_{i+1} \mid \cdots \mid x_n] \\ &\qquad + \cdots + (-1)^{n+1} [y \mid x_1 \mid \cdots \mid x_{n-1}] x_n \Big\} \\ &= |\pi|[x_1 \mid \cdots \mid x_n] - s \circ \partial([x_1 \mid \cdots \mid x_n]). \end{aligned}$$

We have used the fact that $\sum_{y\in\pi}[yx_1 \mid x_2 \mid \cdots \mid x_n] = \sum_{y\in\pi}[y \mid x_2 \mid \cdots \mid x_n]$. If $\sum_j [x_{1j} \mid \cdots \mid x_{nj}] \otimes m_j$ represents a homology class in $H_n(\pi, M)$, then

$$\partial \circ s\Big(\sum\nolimits_j [x_{1j} \mid \cdots \mid x_{nj}] \otimes m_j\Big) = |\pi| \sum\nolimits_j [x_{1j} \mid \cdots \mid x_{nj}] \otimes m_j$$

and so $|\pi|$ times any homology class in $H_n(\pi, M)$ is zero. □

Particular groups may have smaller resolutions. For $\pi = \mathbb{Z}/m\mathbb{Z}$, the cyclic group of order m, there is a very small resolution:Let t denote a generator of π, thought of as a multiplicative group. Let W_i denote the free $\mathbb{Z}\pi$-module on a single generator w_i. There is an acyclic complex

$$\cdots \to W_{2n} \xrightarrow{N} W_{2n-1} \xrightarrow{T} \cdots \xrightarrow{T} W_2 \xrightarrow{N} W_1 \xrightarrow{T} W_0 \xrightarrow{\varepsilon} \mathbb{Z} \to 0$$

where $T(w_{2n-1}) = tw_{2n-2} - w_{2n-2}$ (trace), and $N(w_{2n}) = w_{2n-1} + tw_{2n-1} + \cdots + t^{m-1}w_{2n-1}$ (norm). This resolution allows us to compute $H_i(\mathbb{Z}/m\mathbb{Z})$ immediately from the complex $W_\bullet \otimes_{\mathbb{Z}\pi} \mathbb{Z}$ which takes the form

$$\cdots \xrightarrow{\times m} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{\times m} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{\times m} \mathbb{Z}.$$

Thus $H_{2i+1}(\mathbb{Z}/m\mathbb{Z}) = \{0\}$ and $H_{2i}(\mathbb{Z}/m\mathbb{Z}) = \mathbb{Z}/m\mathbb{Z}$ for all $i \geq 0$.

We close this discussion with a useful lemma.

Lemma 8^{bis}.8. *If M is a free left $\mathbb{Z}\pi$-module, then $H_i(\pi, M) = \{0\}$ for $i > 0$ and $H_0(\pi, M) = M_\pi$. If M is a trivial left $\mathbb{Z}\pi$-module that is free over $\mathbb{Z}$, then $H_i(\pi, M) \cong H_i(\pi) \otimes M$.*

PROOF: The assertion about a free module follows simply from the properties of $\mathrm{Tor}_*^{\mathbb{Z}\pi}(\mathbb{Z}, M)$. For trivial modules we can write $\mathtt{B}_\bullet \otimes_{\mathbb{Z}\pi} M = (\mathtt{B}_\bullet)_\pi \otimes M$ and the result follows. □

The Cartan-Leray spectral sequence

Suppose (X, x_0) is a connected, pointed space on which a **group** π **acts freely and properly**, that is,

(1) for all $x \in X$, the subgroup $G_x = \{g \in G \mid gx = x\}$ is trivial;
(2) every point $x \in X$ has a neighborhood U such that $gU \cap U = \emptyset$ for all $g \in \pi$, $g \neq 1$.

For example, if (X, x_0) is a connected, locally simply-connected space, then the fundamental group, $\pi = \pi_1(X, x_0)$ acts freely and properly on $\tilde{X}$, the universal covering space of X.

Suppose π acts freely and properly on X. For any abelian group G, it is a classical result that $C_*(X/\pi; G) \cong C_*(X; G)_\pi$, where $C_*(\ ; G)$ denotes the singular chains with coefficients in G and $C_*(X;G)$ is a π-module by viewing each $a \in \pi$ as a mapping $a\colon X \to X$. Moreover, $C_i(X) = C_i(X;\mathbb{Z})$ is a free $\mathbb{Z}\pi$-module for all i.

Let $F_\bullet \to \mathbb{Z} \to 0$ denote a free, right, $\mathbb{Z}\pi$-module resolution of $\mathbb{Z}$ as a trivial π-module. Consider the double complex $\mathcal{C}_{\bullet,*}$ given by

$$\mathcal{C}_{p,q} = F_p \otimes_{\mathbb{Z}\pi} C_q(X), \quad \partial_F \otimes 1 + (-1)^q 1 \otimes \partial_X.$$

When we filter $\mathcal{C}_{\bullet,*}$ row-wise, we get $E^0_{*,p} = F_\bullet \otimes_{\mathbb{Z}\pi} C_p(X)$, $d^0 = \partial_F \otimes 1$, which is the complex computing $H_*(\pi, C_p(X))$. However, $C_p(X)$ is a free $\mathbb{Z}\pi$-module and so the E^1-term is concentrated in the 0-column where we find $H_0(\pi, C_p(X)) = C_p(X)_\pi = C_p(X/\pi)$. By the appropriate version of Theorem 2.15, $d^1 = 1 \otimes_\pi \partial_X = \partial_{X/\pi}$ and so the spectral sequence collapses at E^2 to $H_*(X/\pi)$.

When we filter by $\mathcal{C}_{\bullet,*}$ column-wise, we get $E^0_{p,*} = F_p \otimes_{\mathbb{Z}\pi} C_*(X)$, $d^0 = 1 \otimes \partial_X$. Viewing F_p as an extended module of the form $F_p = A_p \otimes \mathbb{Z}\pi$ with A_p a free abelian group, we get the identification

$$F_p \otimes_{\mathbb{Z}\pi} C_*(X) = A_p \otimes C_*(X)$$

and so $E^1_{p,*} = A_p \otimes H_*(X) = F_p \otimes_{\mathbb{Z}\pi} H_*(X)$. This is the complex computing $H_*(\pi, H_*(X))$ and so we have proved the following result of [Cartan-Leray49].

Theorem 8^{bis}.9 (the Cartan-Leray spectral sequence). *If X is a connected space on which the group π acts freely and properly, then there is a spectral sequence of first quadrant, homological type, with*

$$E^2_{p,q} \cong H_p(\pi, H_q(X))$$

and converging strongly to $H_*(X/\pi)$.

More generally, we can use homology with coefficients in an abelian group G taken as a trivial π-module. For the case of the fundamental group acting on the universal cover, the spectral sequence has $E^2_{p,q} \cong H_p(\pi, H_q(\tilde{X}))$, where $\pi = \pi_1(X, x_0)$ and converges to $H_*(X)$. From this spectral sequence we prove a theorem that relates the fundamental group and its homology groups to the higher homology and homotopy groups of a space. The case of $n = 1$ was first proved by [Hopf42] using other methods. This paper launched the study of the homology and cohomology of groups.

Theorem 8^{bis}.10. *Suppose (X, x_0) is a connected, locally simply-connected space whose universal covering space $\tilde{X}$ is n-connected. Let π denote $\pi_1(X, x_0)$. Then there are isomorphisms $H_i(X) \cong H_i(\pi)$ for $1 \le i \le n$, and an exact sequence*

$$H_{n+2}(X) \to H_{n+2}(\pi) \to (H_{n+1}(\tilde{X}))_\pi \to H_{n+1}(X) \to H_{n+1}(\pi) \to 0.$$

PROOF: Since $H_i(\tilde{X}) = \{0\}$ for $1 \leq i \leq n$, we have a big hole in the Cartan-Leray spectral sequence converging to $H_*(X)$. The theorem follows from interpreting the lowest degree information. The isomorphisms $H_i(X) = H_i(\tilde{X}/\pi) \cong H_i(\pi)$ for $1 \leq i \leq n$ follow because there are no differentials involved. The first place there is a possible differential is $d^{n+1}\colon E^2_{n+2,0} \to E^2_{0,n+1}$. This leads to the following short exact sequences:

$$H_{n+2}(X) \to E^\infty_{n+2,0} \to 0$$
$$0 \to E^\infty_{n+2,0} \to H_{n+2}(\pi) \xrightarrow{d^{n+1}} (H_{n+1}(\tilde{X}))_\pi \to E^\infty_{0,n+1} \to 0$$
$$0 \to E^\infty_{0,n+1} \to H_{n+1}(X) \to E^\infty_{n+1,0} \to 0.$$

Splicing these sequences together (as in Example 5.D) and substituting $H_{n+1}(\pi)$ for $E^2_{n+1,0} = E^\infty_{n+1,0}$ we get the desired exact sequence. □

Since $\pi_i(X) \cong \pi_i(\tilde{X})$ for $i \geq 2$, we can substitute $\pi_{n+1}(X)$ for the term in the middle of the exact sequence of Theorem 8^{bis}.10 by the Hurewicz theorem. There is a natural epimorphism of a π-module onto its coinvariants, so we can truncate the short exact sequence to obtain another exact sequence

$$\pi_{n+1}(X) \to H_{n+1}(X) \to H_{n+1}(\pi) \to 0.$$

When $n = 1$, there is no restriction on X except that it have a universal covering space. We conclude from the theorem that $H_1(X) \cong H_1(\pi)$ and so $H_1(\pi) \cong \pi/[\pi,\pi]$ follows from Poincaré's classical isomorphism. We also get the **short exact sequence of Hopf**:

$$\pi_2(X) \to H_2(X) \to H_2(\pi) \to 0.$$

For an **aspherical space**, that is, a space X whose universal cover has trivial higher homotopy groups, the integral homology of X is determined by its fundamental group, $H_i(X) \cong H_i(\pi_1(X, x_0))$ for all i. Examples of aspherical spaces are the Eilenberg-Mac Lane spaces $K(\pi, 1)$. The study of the homology of groups was one of the motivations for [Eilenberg-Mac Lane53] to introduce the spaces $K(\pi, n)$. [Kan-Thurston76] reversed the process of studying groups using spaces by showing that for any pointed, connected space (X, x_0) there is a group GX and a mapping $t_X\colon K(GX, 1) \to X$ that induces an integral homology isomorphism.

Using the same filtration that leads to the Cartan-Leray spectral sequence, we can investigate systems of local coefficients further. In particular, for X a connected space and $\mathcal{G}$ a system of groups on X, we restrict our attention to the reduced homology. Consider the group of the reduced chains $\tilde{C}_q(X; \mathcal{G})$ given by sums of expressions $g \otimes u$ where $u\colon (\Delta^q, (\Delta^q)^{(0)}) \to (X, x_0)$ and $g \in G_{x_0}$, where the singular simplex u sends all its vertices to the basepoint x_0, and g

lies in the group over the basepoint. By the properties of homology groups with coefficients, the study of $H_q(X;\mathcal{G})$ may be carried out using $\tilde{C}_*(X;\mathcal{G})$. We quote here a result of [Eilenberg47] and refer the reader to [Whitehead, GW78, VI.3] for a proof.

Theorem 8

Theorem 8^{bis}.11. *If (X,x_0) is a pointed, path-connected space and $\mathcal{G}$ a bundle of groups over X, then $H_q(X;\mathcal{G})$ is isomorphic to the homology of the complex $G_0 \otimes_\pi C_*(\tilde{X})$ where $G_0 = G_{x_0}$, $\pi = \pi_1(X,x_0)$ and $\tilde{X}$ is the universal covering space of X together with its action of π.*

The proof follows by a direct comparison. We note a useful corollary. Since $\tilde{X}$ is one-connected, the complex $C_*(\tilde{X})$ up to degree two is acyclic and free over π. Thus we could use this complex as part of a free resolution of $\mathbb{Z}$ over π and so identify $H_1(X;\mathcal{G})$ with $H_1(\pi_1(X);G_0)$. This extends Proposition 8^{bis}.4 and the identification of homology groups with local coefficients to include $q=1$. The homology of the complex $G_0 \otimes_\pi C_*(\tilde{X})$ was termed the **equivariant homology of** X by [Eilenberg47] and it represents one of the basic functors in the study of equivariant homotopy theory.

The Lyndon-Hochschild-Serre spectral sequence

The Leray-Serre spectral sequence expresses the relation between the total space of a fibration and its base and fibre. In the realm of groups, a “fibration” is an extension of the form

$$1 \to H \longrightarrow \pi \longrightarrow Q \to 1,$$

where H is normal in π and $Q \cong \pi/H$. The “total space” π is the extension of the “base” Q and “fibre” H. There is a corresponding spectral sequence relating the homology of a group to a normal subgroup and associated quotient.

To describe the spectral sequence we observe that the quotient group Q acts on the homology of H. Let $F_\bullet \xrightarrow{\varepsilon} \mathbb{Z} \to 0$ be a free right $\mathbb{Z}\pi$-module resolution of $\mathbb{Z}$. Then it is also a free right $\mathbb{Z}H$-module resolution. If M is a left π-module, it likewise is a left H-module by restriction. If $g \in \pi$, then define $(g)_*\colon F_\bullet \otimes_{\mathbb{Z}H} M \to F_\bullet \otimes_{\mathbb{Z}H} M$ by $(g)_*(x \otimes m) = xg^{-1} \otimes gm$. It follows formally that $(g)_*$ commutes with the differential on $F_\bullet \otimes_{\mathbb{Z}H} M$ and so $(g)_*$ induces a homomorphism $(g)_*\colon H_*(H,M) \to H_*(H,M)$. From the definition of the tensor product over $\mathbb{Z}H$, $(h)_* = \mathrm{id}$ for all $h \in H$ and so this action of π on $H_*(H,M)$ induces an action of π/H on $H_*(H,M)$. With this bit of structure in place we construct a spectral sequence associated to an extension of groups.

Theorem 8^{bis}.12 (the Lyndon-Hochschild-Serre spectral sequence). *Let $1 \to H \to \pi \to Q \to 1$ be a group extension. Suppose M is a module over π. Then there is a first quadrant spectral sequence with*

$$E^2_{p,q} \cong H_p(Q, H_q(H, M)),$$

and converging strongly to $H_*(\pi, M)$.

PROOF: We begin with some elementary algebraic observations. When we take the coinvariants of a π-module M with respect to the action of H, we make the action of H trivial on M_H. Thus π/H acts on M_H by $gH \cdot (m + \Gamma^2_H M) = gm + \Gamma^2_H M$, where $\Gamma^2_H M$ is the submodule of M generated by elements $hm - m$ for $h \in H$ and $m \in M$.

We next consider the coinvariants of M_H via the $Q = \pi/H$ action and prove the following formula:

$$(M_H)_Q \cong M_\pi.$$

This follows from the relation $\mathbb{Z}Q \otimes_{\mathbb{Z}\pi} M \cong M_H$ by using the isomorphism $\mathbb{Z} \otimes_{\mathbb{Z}\pi} M \cong \mathbb{Z} \otimes_{\mathbb{Z}Q} \mathbb{Z}Q \otimes_{\mathbb{Z}\pi} M$.

Consider the mapping $\phi\colon M_H \to \mathbb{Z}Q \otimes_{\mathbb{Z}\pi} M$ given by $m \mapsto 1 \otimes m$. Since $1 \otimes m = 1 \otimes h^{-1}hm = 1 \cdot h \otimes hm = 1 \otimes hm$, we see that ϕ is defined on M_H. An inverse to ϕ is the mapping $\psi\colon gH \otimes m \mapsto gm + \Gamma^2_H M$. That ψ is well-defined follows from the fact that M_H is a Q-module.

We put these elementary observations to work and suppose that $F_\bullet \to \mathbb{Z} \to 0$ is a free, right, $\mathbb{Z}\pi$-module resolution of $\mathbb{Z}$. Consider the complex computing $H_*(\pi, M)$, that is, $F_\bullet \otimes_{\mathbb{Z}\pi} M$. Since $F_\bullet \otimes_{\mathbb{Z}} M$ is a $\mathbb{Z}\pi$-module by the diagonal action ($g \cdot (x \otimes m) = xg^{-1} \otimes gm$), it is a simple exercise to show that

$$(F_\bullet \otimes_{\mathbb{Z}} M)_\pi \cong F_\bullet \otimes_{\mathbb{Z}\pi} M,$$

and so $F_\bullet \otimes_{\mathbb{Z}\pi} M \cong ((F_\bullet \otimes M)_H)_Q$.

We also have that $H(F_\bullet \otimes_{\mathbb{Z}H} M) = H_*(H, M)$. We want to compute the coinvariants of the action of Q on these homology groups. Let $\tilde{F}_\bullet \to \mathbb{Z} \to 0$ be a free right $\mathbb{Z}Q$-module resolution of $\mathbb{Z}$. Form the double complex

$$C_{\bullet *} = \tilde{F}_\bullet \otimes_{\mathbb{Z}Q} (F_* \otimes M)_H,$$

where Q acts on $(F_* \otimes M)_H$ by the diagonal action. When we filter columnwise, we get the complex $(\tilde{F}_\bullet \otimes_{\mathbb{Z}Q} (F_q \otimes_{\mathbb{Z}H} M), d \otimes 1)$. However, treating F_q as an extended $\mathbb{Z}\pi$-module (on the left via the antiautomorphism $g \mapsto g^{-1}$ on $\mathbb{Z}\pi$), we can write $F_q = \mathbb{Z}\pi \otimes A_q$ for some free $\mathbb{Z}$-module A_q. It follows that

$$F_q \otimes_{\mathbb{Z}H} M = \mathbb{Z} \otimes_{\mathbb{Z}H} (F_q \otimes M) = \mathbb{Z} \otimes_{\mathbb{Z}H} (\mathbb{Z}\pi \otimes A_q \otimes M) = \mathbb{Z}Q \otimes A_q \otimes M,$$

because $(\mathbb{Z}\pi)_H = \mathbb{Z}Q$. Thus we have $E^0_{\bullet,q} \cong \tilde{F}_\bullet \otimes_{\mathbb{Z}Q} \mathbb{Z}Q \otimes A_q \otimes M$ and so $E^1_{p,q} \cong H_p(Q, (F_q \otimes_{\mathbb{Z}H} M)) = \{0\}$ if $p > 0$ and if $p = 0$ $E^1_{0,q}$ is isomorphic to $(F_q \otimes_{\mathbb{Z}H} M)_Q = (F_q \otimes_{\mathbb{Z}\pi} M)$. Thus $E^2_{0,*} = H_*(\pi, M)$, and the double complex has total homology given by $H_*(\pi, M)$.

When we filter row-wise, we have $E^0_{p,*} = \tilde{F}_p \otimes_{\mathbb{Z}Q} (F_* \otimes_{\mathbb{Z}H} M)$ and $d^0 = 1 \otimes d$. We argue similarly and write $\tilde{F}_p$ as an extended right $\mathbb{Z}Q$-module, $\tilde{A}_p \otimes \mathbb{Z}Q$, to get $E^0_{p,*} \cong \tilde{A}_p \otimes (F_* \otimes_{\mathbb{Z}H} M)$, $d^0 = 1 \otimes d$ and so $E^1_{p,*} \cong \tilde{A}_p \otimes H_*(H, M) \cong \tilde{F}_p \otimes_{\mathbb{Z}Q} H_*(H, M)$ and $d^1 = d \otimes 1$. Thus $E^2_{p,q} \cong H_p(Q, H_q(H, M))$. □

The first calculations of some of the relations implied by this spectral sequence appeared in Chicago Ph.D. thesis of [Lyndon48] without the benefit of the structure made apparent in the work of [Leray46]. The cohomology version of the spectral sequence of Theorem 8^{bis}.12 first appeared in the Comptes Rendues note of [Serre50']. [Hochschild-Serre53] introduced the spectral sequence of Theorem 8^{bis}.12 along with a formalism that allowed analogous constructions for Lie algebras. Their work was based on a different filtration on the cochain complex whose homology gives the cohomology of a group. Another point of view that gives rise to this spectral sequence is due to [Grothendieck57]. The composition of functors spectral sequence (see Chapter 11) for the composition of the coinvariants of the H action followed by the coinvariants of the π/H action gives another construction of the Lyndon-Hochschild-Serre spectral sequence. The fact that these alternative constructions give isomorphic spectral sequences from the E_2-term is due to [Beyl81].

A simple example of the use of the spectral sequence is the case of the extension associated to a Sylow p-subgroup when it is normal in a finite group.

Proposition 8^{bis}.13. *Suppose π is a finite group and P is a normal Sylow p-subgroup. Then*

$$H_i(\pi, \mathbb{F}_p) \cong H_i(P, \mathbb{F}_p)_Q,$$

where π acts on $\mathbb{F}_p$ trivially and $Q = \pi/P$.

PROOF: The spectral sequence associated to the extension $1 \to P \to \pi \to Q \to 1$ has E^2-term given by $E^2_{p,q} \cong H_p(Q, H_q(P, \mathbb{F}_p))$. Since the order of Q is relatively prime to p and p times any element in $H_q(P, \mathbb{F}_p)$ is zero, we conclude from Proposition 8^{bis}.7 that $E^2_{p,q} \cong \{0\}$ for $p > 0$. Thus the spectral sequence collapses to a single column given by $E^2_{0,q} \cong H_q(P, \mathbb{F}_p)_Q$ and the result follows. □

We next consider the “lower left-hand corner” of the spectral sequence. Theorems 8^{bis}.14 and 8^{bis}.16 were first obtained by [Stallings65] and [Stammbach66].

Theorem 8^{bis}.14. *To an extension of groups* $1 \to H \to \pi \to Q \to 1$*, there is an exact sequence*

$$H_2(\pi) \to H_2(Q) \to H/[\pi, H] \to H_1(\pi) \to H_1(Q) \to 0.$$

PROOF: This is simply the five-term exact sequence from splicing together the differential d^2 with the associated graded filtration for $H_1(\pi)$ (Example I.1) as in the diagram:

$$\begin{tikzcd}
 & H_2(\pi) \arrow[d] \arrow[dr, dashed] & & & 0 \arrow[d] & \\
0 \arrow[r] & E^\infty_{2,0} \arrow[r] \arrow[d] & E^2_{2,0} \arrow[r, "d^2"] \arrow[d, equal] & E^2_{0,1} \arrow[r] \arrow[d, equal] \arrow[dr, dashed] & E^\infty_{0,1} \arrow[r] \arrow[d] & 0 \\
 & 0 & H_2(Q) & (H_1(H))_Q & H_1(\pi) \arrow[d] & \\
 & & & & E^\infty_{1,0} \arrow[r, equal] \arrow[d] & H_1(Q) \\
 & & & & 0 &
\end{tikzcd}$$

To complete the proof we make the identification $(H_1(H))_Q \cong H/[\pi, H]$. Recall that $H_1(H) = H/[H, H]$ and that Q acts on this group by conjugation. The conjugation action of π on itself has coinvariants $\pi/[\pi, \pi] = H_1(\pi)$ and it is the residue of this action that remains as Q acts on $H_1(H)$. The Q-coinvariants of the conjugation action on $H/[H, H]$ are given by $H/[\pi, H]$ and so the theorem is proved. □

To extend this theorem further we recall the lower central series of a group—a version of this series will play an important role in the study of nilpotent groups and spaces.

Definition 8^{bis}.15. *Let* π *denote a group. The* **lower central series** *of* π *is the family of subgroups defined inductively,*

$$\cdots \subset \Gamma^r\pi \subset \Gamma^{r-1}\pi \subset \cdots \subset \Gamma^2\pi \subset \pi$$
$$\Gamma^2\pi = [\pi, \pi], \qquad \Gamma^r\pi = [\pi, \Gamma^{r-1}\pi].$$

To the lower central series of π *we associate the* **completion** *of* π*:*

$$\widehat{\pi} = \lim_{\leftarrow r} \pi/\Gamma^r\pi.$$

We can extend the lower central series to higher ordinals by letting $\Gamma^\alpha\pi = [\pi, \Gamma^\beta\pi]$ when $\alpha = \beta + 1$ and $\Gamma^\alpha\pi = \bigcap_{\beta<\alpha} \Gamma^\beta\pi$ when α is a limit ordinal.

Theorem 8^{bis}.16. *Suppose $\phi\colon \pi \longrightarrow \pi'$ is a homomorphism of groups that induces an isomorphism $H_1(\phi)\colon H_1(\pi) \to H_1(\pi')$ and a surjection $H_2(\phi)\colon H_2(\pi) \twoheadrightarrow H_2(\pi')$. Then the induced mapping $\bar{\phi}\colon \pi/\Gamma^r\pi \to \pi'/\Gamma^r\pi'$ is an isomorphism for all positive integers r. Furthermore, ϕ induces an isomorphism of completions $\widehat{\phi}\colon \widehat{\pi} \to \widehat{\pi}'$.*

PROOF: We work by induction. We let $\Gamma^1\pi = \pi$ and $\Gamma^1\pi' = \pi'$, and so the induction begins trivially. Next consider the extension $1 \to \Gamma^r\pi \longrightarrow \pi \longrightarrow \pi/\Gamma^r\pi \to 1$ and similarly for π'. By Theorem 8^{bis}.14 we have short exact sequences

$$\begin{tikzcd}
H_2(\pi) \arrow[r] \arrow[d, "\phi_*"'] & H_2(\pi/\Gamma^r\pi) \arrow[r] \arrow[d, "\tilde{\phi}_*"'] & \Gamma^r\pi/[\pi,\Gamma^r\pi] \arrow[r] \arrow[d] & H_1(\pi) \arrow[r] \arrow[d, "\phi_*"'] & H_1(\pi/\Gamma^r\pi) \arrow[r] \arrow[d, "\tilde{\phi}_*"'] & 0 \\
H_2(\pi') \arrow[r] & H_2(\pi'/\Gamma^r\pi') \arrow[r] & \Gamma^r\pi'/[\pi',\Gamma^r\pi'] \arrow[r] & H_1(\pi') \arrow[r] & H_1(\pi'/\Gamma^r\pi') \arrow[r] & 0
\end{tikzcd}$$

The induction hypothesis together with the Five-lemma implies that ϕ induces an isomorphism $\Gamma^r\pi/[\pi,\Gamma^r\pi] \longrightarrow \Gamma^r\pi'/[\pi',\Gamma^r\pi']$. Furthermore, by definition, $\Gamma^r\pi/[\pi,\Gamma^r\pi] = \Gamma^r\pi/\Gamma^{r+1}\pi$, and so there is a short exact sequence

$$1 \to \Gamma^r\pi/\Gamma^{r+1}\pi \longrightarrow \pi/\Gamma^{r+1}\pi \longrightarrow \pi/\Gamma^r\pi \to 1$$

and similarly for π'. The homomorphism ϕ induces a morphism of short exact sequences and the Five-lemma implies that ϕ induces an isomorphism of $\pi/\Gamma^{r+1}\pi$ with $\pi'/\Gamma^{r+1}\pi'$. □

Applications of the Lyndon-Hochschild-Serre spectral sequence abound. We refer the reader to the book-length treatments of the cohomology of groups by [Brown, K82], [Evens93], [Weibel94], [Thomas86], [Benson91], and [Adem-Milgram94] for more of the algebraic details, applications, and structure of this spectral sequence. [Huebschmann81, 91] has studied the differentials in the cohomology Lyndon-Hochschild-Serre spectral sequence and made successful use of this information to compute the cohomology of some important classes of groups. Some of the exercises at the end of the chapter expose the algebraic possibilities afforded by this tool.

§8^{bis}.3 Nilpotent spaces and groups

When a group fails to be abelian, the lower central series is a composition series whose consecutive quotients are abelian. One direction to generalize the category of abelian groups is to consider groups whose lower central series is bounded in length. A group π is said to be **nilpotent** of **nilpotence class** c if $c = \max\{r \mid \Gamma^r\pi \neq \{1\}\}$ is finite. Thus $\Gamma^{c+k}\pi = [\pi, \Gamma^{c+k-1}\pi] = \{1\}$ for $k \geq 1$.

If we view π as a group on which π acts by conjugation, then $\Gamma^2\pi = [\pi,\pi]$ is the smallest normal subgroup for which the quotient has an induced trivial action. We generalize this idea to the action of a group π on a module M.

Definition 8^{bis}.17. *The* **lower central series** *associated to the action of a group π on a module M is the series of submodules*

$$\cdots \subset \Gamma_\pi^r M \subset \Gamma_\pi^{r-1} M \subset \cdots \subset \Gamma_\pi^2 M \subset M$$

where $\Gamma_\pi^2 M$ is the $\mathbb{Z}\pi$-submodule of M generated by elements of the form $am - m$ for $a \in \pi$ and $m \in M$, and $\Gamma_\pi^r M = \Gamma_\pi^2(\Gamma_\pi^{r-1} M)$. We say that π **acts nilpotently on** *M of* **nilpotence class** *c if $c = \max\{r \mid \Gamma_\pi^r M \neq \{0\}\}$ is finite.*

To a group or to a group action on a module, we associate various completion functors:

$$\widehat{\pi} = \lim_{\leftarrow r} \pi/\Gamma^r\pi, \qquad \widehat{M} = \lim_{\leftarrow r} M/\Gamma_\pi^r M.$$

By the universal property of the inverse limit there are canonical homomorphisms $i\colon \pi \to \widehat{\pi}$ and $i\colon M \to \widehat{M}$. Define

$$\begin{aligned} \Gamma^\infty \pi &= \ker(i\colon \pi \to \widehat{\pi}) & \Gamma_\pi^\infty M &= \ker(i\colon M \to \widehat{M}) \\ \Gamma'_\infty \pi &= \operatorname{coker}(i\colon \pi \to \widehat{\pi}) & \Gamma'_\infty M &= \operatorname{coker}(i\colon M \to \widehat{M}). \end{aligned}$$

By Lemma 3.10, $\Gamma_\pi^\infty M = \bigcap_r \Gamma_\pi^r M$.

Notice that, for a nilpotent group or nilpotent group action, i is an isomorphism and the functors Γ^∞ and Γ'_∞ vanish. Before turning to some examples, we introduce one more functor. A submodule N of a π-module M is said to **π-perfect** if $\Gamma_\pi^2 N = N$. If we take the family $\mathcal{P}_M$ of all π-perfect submodules of M, then we can define

$$\Gamma M = \text{ submodule generated by the union of the family } \mathcal{P}_M.$$

We leave it to the reader to show that ΓM is also π-perfect; ΓM is the **maximal π-perfect submodule** of M as it contains all π-perfect submodules. By construction, $\Gamma M \subset \Gamma_\pi^\infty M$.

A nice example of a nonnilpotent group action is given by our example $\pi_n(S^{2m} \times \mathbb{R}P^{2n})$. Here the generator of $\pi_1(S^{2m} \times \mathbb{R}P^{2n}) = \mathbb{Z}/2\mathbb{Z}$ acts on $\mathbb{Z}$ by $1 \mapsto -1$. The lower central series for this action is seen to be

$$\cdots \subset 2^r\mathbb{Z} \subset 2^{r-1}\mathbb{Z} \subset \cdots \subset 2\mathbb{Z} \subset \mathbb{Z}$$

and so the action is nonnilpotent. The completion of $\mathbb{Z}$ is the group of 2-adic integers, $\Gamma_{\mathbb{Z}/2\mathbb{Z}}^\infty \mathbb{Z} = \{0\} = \Gamma\mathbb{Z}$. Notice that $\Gamma'_\infty \mathbb{Z}$ is an uncountable group.

If M is a module over the group π, then we develop another expression for the terms in the lower central series for M. Let $I\pi$ denote the kernel of the augmentation $\varepsilon\colon \mathbb{Z}\pi \to \mathbb{Z}$, given by $\varepsilon(\sum_{g\in\pi} n_g g) = \sum_{g\in\pi} n_g$.

Lemma 8^{bis}.18. *$\Gamma^2_\pi M = I\pi \cdot M$, that is, $\Gamma^2_\pi M$ is the submodule of M generated by expressions of the form xm with $x \in I\pi$ and $m \in M$.*

PROOF: Since $\Gamma^2_\pi M$ is the submodule of M generated by $am - m$ with $a \in \pi$ and $m \in M$, we can write

$$am - m = (a-1)m \in I\pi \cdot M.$$

Thus $\Gamma^2_\pi M \subset I\pi \cdot M$. To obtain the reverse inclusion, take any element $a = \sum_{g\in\pi} n_g g$ in $\mathbb{Z}\pi$ and consider $\sum_{g\in\pi} n_g g - \sum_{g\in\pi} n_g 1$. This is an element in $I\pi$. For any element m in M we can now write

$$\Big(\sum\nolimits_{g\in\pi} n_g(g-1)\Big) \cdot m = \sum\nolimits_{g\in\pi} n_g(gm - m) \in \Gamma^2_\pi M.$$

It remains to show that every element of $I\pi$ has the form $\sum_{g\in\pi} n_g(g-1)$. To see this, write $a = \sum_{g\in\pi} n_g g = n_1 1 + \sum_{g\in\pi, g\neq 1} n_g g$. When $a \in I\pi$, $\sum_{g\in\pi,g\neq 1} n_g = -n_1$, and so we can write $a = \sum_{g\in\pi,g\neq1} n_g g - \sum_{g\in\pi,g\neq1} n_g 1$. Thus $I\pi \cdot M \subset \Gamma^2_\pi M$. □

Corollary 8^{bis}.19. *If $0 \to K \to M \to Q \to 0$ is a short exact sequence of π-modules and π-module homomorphisms, then M is a nilpotent π-module if and only if K and Q are.*

We next introduce the topological category of interest.

Definition 8^{bis}.20. *A pointed connected CW-complex (X, x_0) is said to be* **nilpotent** *if $\pi_1(X, x_0)$ is a nilpotent group that acts nilpotently on $\pi_n(X, x_0)$ for all $n \geq 2$.*

The category of all nilpotent spaces together with continuous mappings contains all simply-connected spaces, all H-spaces (Corollary 8^{bis}.3), and the Eilenberg-Mac Lane spaces $K(\pi, 1)$ for π nilpotent. As we will soon see, this category is also closed under certain constructions of importance in homotopy theory.

There are spaces that are not nilpotent. Some general constructions are possible: (1) closed surfaces of genus greater than one; (2) the wedge product $X \vee Y$ of any two non-simply connected spaces X and Y; (3) if X has fundamental group π and there is an epimorphism $\pi \to G$ with G finite and G acts nontrivially on the rational homology of the covering space of X corresponding to the kernel of $\pi \to G$, then X is not nilpotent.

We begin our investigation of some of these constructions with a useful analogue of Corollary 8^{bis}.19.

Proposition 8^{bis}.21. *If $F \hookrightarrow E \xrightarrow{p} B$ is a fibration with F connected and E nilpotent, then F is nilpotent.*

PROOF: We can apply Lemma 8^{bis}.18 inductively to show that $\Gamma_\pi^r M = (I\pi)^{r-1} \cdot M$. Suppose that the nilpotence class of $\pi_n(E)$ as a module over $\pi = \pi_1(E)$ is c. Then $\Gamma_\pi^{c+1}\pi_n(E) = \{0\}$, and so $(I\pi)^c \cdot \pi_n(E) = \{0\}$.

Suppose that $\omega \in \pi_1(F)$ and $\alpha \in \pi_n(F)$ for $n \geq 2$. Then $\omega \cdot \alpha = (i_*\omega) \cdot \alpha$ in terms of the $\pi_1(E)$-module structure on $\pi_n(F)$. Suppose that $\omega \in (I\pi)^c$. Since i_* is a $\pi_1(E)$-module homomorphism, $i_*(\omega \cdot \alpha) = (i_*\omega) \cdot (i_*\alpha) = 0$. Thus $\omega \cdot \alpha = \partial\beta$ for some $\beta \in \pi_{n+1}(B)$. Suppose that $\eta \in \pi_1(F)$. Then $i_*\eta \in \pi_1(E)$ and

$$\partial((i_*\eta - 1) \cdot \beta) = (i_*\eta - 1)(\partial\beta) = (i_*\eta - 1)(\omega \cdot \alpha) = (\eta - 1)(\omega \cdot \alpha).$$

However, $\pi_1(E)$ acts on $\pi_{n+1}(B)$ via p_*, and so $(i_*\eta - 1) \cdot \beta = (p_* i_* \eta - 1) \cdot \beta = -\beta$. Thus $\eta \cdot \omega \cdot \alpha = 0$. This shows that $(I\pi_1(F))^{c+1} \cdot \pi_n(F) = \{0\}$ and so the nilpotence class of $\pi_n(F)$ over $\pi_1(F)$ is less than or equal to $c + 1$.

When $n = 1$ the argument for $\pi_1(F)$ acting on itself by conjugation is similar and left to the reader. □

We next present one of the first results on nilpotent spaces, due to [Whitehead, GW54] in a paper in which the nilpotence class is related to the Lusternik-Schnirelmann category (for the purposes of proving the Jacobi identity for the Whitehead product!).

Theorem 8^{bis}.22. *Suppose (A, a_0) is a finite, connected, pointed CW-complex and (X, x_0) is a connected, pointed CW-complex. Suppose $f\colon (A, a_0) \to (X, x_0)$ is a fixed pointed map. In the compact-open topology, the space $\operatorname{map}_f((A, a_0), (X, x_0))$ of pointed maps in the component of f is a nilpotent space. Furthermore, if X is nilpotent, then the space of unpointed maps $\operatorname{map}_f(A, X)$ is nilpotent.*

PROOF: We proceed by induction. We assume that $A^{(0)}$, the 0-skeleton of A, consists of a single point, a_0. Then the theorem is seen to be true for $\operatorname{map}_f((a_0, a_0), (X, x_0))$ and $\operatorname{map}_f(A^{(0)}, X)$.

Consider the cofibration $\bigvee_{i=1}^k S^n \longrightarrow A^{(n)} \longrightarrow A^{(n+1)}$ that determines the addition of cells to the next skeleton. Apply the functor $\operatorname{map}_f(—, (X, x_0))$ to obtain a fibration (see §4.3):

$$\begin{aligned} \operatorname{map}_f((A^{(n+1)}, a_0), (X, x_0)) \to \operatorname{map}_{f|}&((A^{(n)}, a_0), (X, x_0)) \\ &\to \operatorname{map}_{f|}((\textstyle\bigvee_{i=1}^k S^n, *), (X, x_0)). \end{aligned}$$

By induction, we assume that $\operatorname{map}_{f|}((A^{(n)}, a_0), (X, x_0))$ is nilpotent. Proposition 8^{bis}.21 finishes the proof. □

A generalized Whitehead Theorem

When does a mapping $f\colon X \to Y$ of connected spaces that induces an isomorphism of homology groups induce an isomorphism of homotopy groups here taken as a family of π_1-modules? An answer to this question would generalize the Whitehead theorem (Theorem 4.5) to non-simply connected spaces. The functors introduced to study the action of a group on a module in fact give the key to the answer. The following theorem of [Dror71] helped to establish the category of nilpotent spaces as an appropriate category for homotopy theory.

Theorem 8^{bis}.23 (the generalized Whitehead theorem). *Let $f\colon X \to Y$ be a map of connected pointed spaces such that $H_*(f)\colon H_*(X) \to H_*(Y)$ is an isomorphism. Then $\pi_*(f)$ is also an isomorphism if, in addition, f satisfies the three conditions:*

(1) $\Gamma^\infty\pi_*(f)\colon \ker(\pi_*(X) \to \widehat{\pi_*(X)}) \to \ker(\pi_*(Y) \to \widehat{\pi_*(Y)})$ *is an epimorphism.*

(2) $\Gamma'_\infty\pi_*(f)\colon \operatorname{coker}(\pi_*(X) \to \widehat{\pi_*(X)}) \to \operatorname{coker}(\pi_*(Y) \to \widehat{\pi_*(Y)})$ *is a monomorphism.*

(3) $\Gamma\pi_*(f)\colon \Gamma\pi_*(X) \to \Gamma\pi_*(Y)$ *is a monomorphism.*

Theorem $8^{\rm bis}$.23 fixes the role of the various limit functors. For example, if X and Y are aspherical spaces and $f\colon X \to Y$ is a map for which $\Gamma\pi_*(f)$ is an isomorphism, then f is a homotopy equivalence. When the functors Γ_π^∞, Γ'_π and Γ all vanish on $\pi_*(X)$, then the completion homomorphism $\pi_*(X) \to \widehat{\pi_*(X)}$ is an isomorphism and the space X is called π-**complete**. When a map between π-complete spaces is a homology isomorphism, it induces an isomorphism of homotopy groups. All nilpotent spaces are π-complete.

Corollary 8^{bis}.24. *If X and Y are connected nilpotent spaces and $f\colon X \to Y$ is a mapping that induces an isomorphism on integral homology, then f induces an isomorphism of homotopy groups as graded modules over their fundamental groups.*

To prove the theorem we sneak up on it inductively. Let S_n denote the collection of statements:

1_n. $\pi_j(f)\colon \pi_j(X) \to \pi_j(Y)$ is an isomorphism of π_1-modules for $0 \le j \le n-1$.

2_n. $H_n(f)\colon H_n(X) \to H_n(Y)$ is an isomorphism and $H_{n+1}(f)$ is an epimorphism.

3_n. $\Gamma^\infty\pi_n(f)\colon \Gamma^\infty_{\pi_1(X)}\pi_n(X) \to \Gamma^\infty_{\pi_1(Y)}\pi_n(Y)$ is an epimorphism.

4_n. $\Gamma'_\infty\pi_n(f)\colon \Gamma'_\infty\pi_n(X) \to \Gamma'_\infty\pi_n(Y)$ is a monomorphism.

5_n. $\Gamma\pi_n(f)\colon \Gamma\pi_n(X) \to \Gamma\pi_n(Y)$ is a monomorphism.

The condition S_1 holds by the assumptions of Theorem 8^{bis}.23. We claim that S_n implies that $\pi_n(f)$ is an isomorphism and hence that S_{n+1} holds. Proving this claim gives us the theorem. We first prove that, among S_n, the statements 1_n and 2_n imply that $\widehat{\pi_n(f)}\colon \widehat{\pi_n(X)} \to \widehat{\pi_n(Y)}$ is an isomorphism. This follows from two remarkable lemmas due to [Dror71], the first of which extends Theorem 8^{bis}.16.

Lemma 8^{bis}.25. *Suppose $\phi\colon M \to M'$ is a homomorphism of modules over a group π. If $H_0(\phi)\colon H_0(\pi, M) \to H_0(\pi, M')$ is an isomorphism and $H_1(\phi)\colon H_1(\pi, M) \to H_1(\pi, M')$ is an epimorphism, then ϕ induces an isomorphism $M/\Gamma_\pi^r M \to M'/\Gamma_\pi^r M'$ for all $r \geq 1$ and hence, an isomorphism $\widehat{\phi}\colon \widehat{M} \to \widehat{M'}$ of completions.*

PROOF: To the short exact sequence of π-modules

$$0 \to \Gamma_\pi^2 M \to M \to M/\Gamma_\pi^2 M \to 0,$$

there is a long exact sequence of homology groups, ending with

$$\begin{aligned} H_1(\pi, M) \to H_1(\pi, M/\Gamma_\pi^2 M) \to H_0(\pi, \Gamma_\pi^2 M) \\ \to H_0(\pi, M) \to H_0(\pi, M/\Gamma_\pi^2 M). \end{aligned}$$

$H_0(\pi, M) = M/\Gamma_\pi^2 M$, and, since $M/\Gamma_\pi^2 M$ has a trivial π-action,

$$H_0(\pi, M/\Gamma_\pi^2 M) \cong H_0(\pi) \otimes M/\Gamma_\pi^2 M \cong \mathbb{Z} \otimes M/\Gamma_\pi^2 M \cong H_0(\pi, M).$$

This shows that the last homomorphism in the exact sequence is the isomorphism of coinvariants $M_\pi \to (M_\pi)_\pi$. The interesting part of the long exact sequence becomes $H_1(\pi, M) \to H_1(\pi, M/\Gamma_\pi^2 M) \to H_0(\pi, \Gamma_\pi^2 M) \to 0$.

Suppose $H_1(\phi)\colon H_1(\pi, M) \to H_1(\pi, M')$ is an epimorphism and that ϕ induces an isomorphism $\phi\colon M/\Gamma_\pi^2 M \to M'/\Gamma_\pi^2 M'$. Then we have the diagram

$$\begin{tikzcd}
H_1(\pi, M) \arrow[r] \arrow[d, two heads] & H_1(\pi, M/\Gamma_\pi^2 M) \arrow[r] \arrow[d, "\cong"'] & \Gamma_\pi^2 M/\Gamma_\pi^2(\Gamma_\pi^2 M) \arrow[r] \arrow[d] & 0 \\
H_1(\pi, M') \arrow[r] & H_1(\pi, M'/\Gamma_\pi^2 M') \arrow[r] & \Gamma_\pi^2 M'/\Gamma_\pi^2(\Gamma_\pi^2 M') \arrow[r] & 0.
\end{tikzcd}$$

The Five-lemma implies an isomorphism $\Gamma_\pi^2 M/\Gamma_\pi^3 M \to \Gamma_\pi^2 M'/\Gamma_\pi^3 M'$. By the same quotient argument as in the proof of Theorem 8^{bis}.16, we see that ϕ induces an isomorphism $M/\Gamma_\pi^3 M \to M'/\Gamma_\pi^3 M'$. The lemma follows by applying the same argument inductively. □

Corollary 8

Corollary 8^{bis}.26. *If $\phi\colon M \to M'$ is a π-module homomorphism, M and M' are nilpotent, and ϕ induces an isomorphism $H_0(\phi)$ and an epimorphism $H_1(\phi)$, then M and M' are isomorphic.*

The next lemma provides another step in proving the generalized Whitehead Theorem.

Lemma 8^{bis}.27. *Suppose X is a connected space and $K(\pi_n(X), n) \hookrightarrow P_nX \to P_{n-1}X$ is the n^{th} fibration in the Postnikov tower for X. Then there is an exact sequence, functorial in X, given by*

$$\begin{aligned} &H_{n+2}(P_nX) \to H_{n+2}(P_{n-1}X) \to H_1(\pi_1(X), \pi_n(X)) \to H_{n+1}(P_nX) \\ &\qquad \to H_{n+1}(P_{n-1}X) \to (\pi_n(X))_\pi \to H_n(X) \to H_n(P_{n-1}X) \to 0 \end{aligned}$$

PROOF: The Leray-Serre spectral sequence for this fibration has E^2-term given by $E^2_{p,q} \cong H_p(P_{n-1}X; \mathcal{H}_q(K(\pi_n(X), n))$, where the action of $\pi = \pi_1(X)$ on $\pi_n(X)$ determines the local coefficients. Since $H_{n+1}(K(\pi_n(X), n)) = \{0\}$ (a consequence of Lemma 6.2) and $K(\pi_n(X), n)$ is $(n-1)$-connected, we get a lacunary E^2-term in bidegrees $(*, i)$ for $i \leq n+1$—there are only two nonzero stripes in bidegrees $(*, 0)$ and $(*, n)$. As in the derivation of the Gysin sequence (Example 1.D) we get short exact sequences

$$\begin{gathered} 0 \to E^\infty_{n+1,0} \to E^2_{n+1,0} \xrightarrow{d^{n+1}} E^2_{0,n} \to E^\infty_{0,n} \to 0 \\ 0 \to E^\infty_{n+2,0} \to E^2_{n+2,0} \xrightarrow{d^{n+1}} E^2_{1,n} \to E^\infty_{1,n} \to 0 \\ 0 \to E^\infty_{0,n} \to H_n(P_nX) \to E^\infty_{n,0} \to 0 \\ 0 \to E^\infty_{1,n} \to H_{n+1}(P_nX) \to E^\infty_{n+1,0} \to 0 \end{gathered}$$

Splicing these together we get

$$\begin{aligned} &H_{n+2}(P_nX) \to H_{n+2}(P_{n-1}X) \to H_1(P_{n-1}X, \mathcal{H}_n(K(\pi_n(X), n))) \\ &\to H_{n+1}(P_nX) \to H_{n+1}(P_{n-1}X) \to H_0(P_{n-1}X; \mathcal{H}_n(K(\pi_n(X), n))) \\ &\qquad\qquad\qquad \to H_n(P_nX) \to H_n(P_{n-1}X) \to 0 \end{aligned}$$

However, from Proposition 8^{bis}.4 and Theorem 8^{bis}.10 we know that

$$H_i(P_{n-1}X; \mathcal{H}_n(K(\pi_n, n))) \cong \begin{cases} (\pi_n(X))_\pi, & i = 0, \\ H_1(\pi, \pi_n(X)), & i = 1. \end{cases}$$

By the definition of a Postnikov tower, we have that $H_n(P_nX) = H_n(X)$. Furthermore, $H_n(X) \twoheadrightarrow H_n(P_{n-1}X)$ because $H_n(P_{n-1}X) = E^2_{n,0} = E^\infty_{n,0}$. The lemma follows after we make these substitutions in the exact sequence. □

We now complete the proof of Theorem $8^{\mathrm{bis}}.23$. Suppose that our map $f\colon X \to Y$ satisfies the conditions S_n. Then f induces a map of Postnikov towers and by naturality of the short exact sequence of Lemma $8^{\mathrm{bis}}.27$ we get a morphism of exact sequences

$$\begin{tikzcd}
H_{n+1}(P_nX) \arrow[r] \arrow[d, "2_n\cdot"', "\text{epi}"] & H_{n+1}(P_{n-1}X) \arrow[r] \arrow[d, "\cong"] & (\pi_n(X))_\pi \arrow[r] \arrow[d, "f_*"'] & H_n(X) \arrow[r] \arrow[d, "2_n\cdot"', "\cong"] & H_n(P_{n-1}X) \arrow[r] \arrow[d, "\cong"] & 0 \\
H_{n+1}(P_nY) \arrow[r] & H_{n+1}(P_{n-1}Y) \arrow[r] & (\pi_n(Y))_\pi \arrow[r] & H_n(Y) \arrow[r] & H_n(P_{n-1}Y) \arrow[r] & 0.
\end{tikzcd}$$

The leftmost horizontal map is seen to be an epimorphism by considering the next stage of the Postnikov tower where we have $H_{n+1}(X) \twoheadrightarrow H_{n+1}(P_nX)$, and similarly for Y. Since $H_{n+1}(f)$ is an epimorphism by 2_n, we get the first vertical epimorphism. By the Five-lemma, $(\pi_n(X))_\pi \to (\pi_n(Y))_\pi$ is an isomorphism. Next consider the other end of the exact sequence:

$$\begin{tikzcd}
H_{n+2}(P_{n-1}X) \arrow[r] \arrow[d, "2_n\cdot"', "\cong"] & H_1(\pi_1(X), \pi_nX) \arrow[r] \arrow[d] & H_{n+1}(P_nX) \arrow[r] \arrow[d, "2_n\cdot"', "\text{epi}"] & H_{n+1}(P_{n-1}X) \arrow[r] \arrow[d, "\cong"] & \pi_n(X)_\pi \arrow[d, "f_*"', "\cong"] \\
H_{n+2}(P_{n-1}Y) \arrow[r] & H_1(\pi_1(Y), \pi_nY) \arrow[r] & H_{n+1}(P_nY) \arrow[r] & H_{n+1}(P_{n-1}Y) \arrow[r] & \pi_n(Y)_\pi.
\end{tikzcd}$$

The Five-lemma implies that $H_1(\pi_1(X), \pi_n(X)) \to H_1(\pi_1(Y), \pi_n(Y))$ is an epimorphism. By Lemma $8^{\mathrm{bis}}.25$ we have that $\pi_n(f)$ induces an isomorphism between $\pi_n(X)/\Gamma_\pi^r\pi_n(X)$ and $\pi_n(Y)/\Gamma_\pi^r\pi_n(Y)$ for all r and hence induces an isomorphism $\widehat{\pi_n(f)}$. Finally we use the remaining conditions of S_n.

There are exact sequences of functors given by

$$0 \to \Gamma^\infty\pi_n \to \pi_n \to \widehat{\pi_n} \to \Gamma'_\infty\pi_n \to 0$$

$$0 \to \Gamma\pi_n \to \pi_n \to \pi_n/\Gamma\pi_n \to 0.$$

The Five-lemma and conditions 3_n, 4_n, and 5_n for $\pi_n(f)$ imply that $\pi_n(f)$ is an epimorphism.

To prove that $\pi_n(f)$ is a monomorphism, we use 5_n, that is, $\Gamma\pi_n(f)$ is a monomorphism. We only need to show that $\pi_nX/\Gamma\pi_n(X) \to \pi_n(Y)/\Gamma\pi_n(Y)$ is a monomorphism. The lower central series has the property that $\Gamma_\pi^r M \subset \Gamma_\pi^{r-1}M$ is always strictly decreasing until it becomes stable. This is because $\Gamma_\pi^r M = \Gamma_\pi^2(\Gamma_\pi^{r-1}M)$. We also know that $\Gamma M \subset \Gamma_\pi^r M$ for all r. In fact, this inclusion extends to r, any transfinite ordinal, as follows: If $\alpha = \beta + 1$ are ordinals, then let $\Gamma_\pi^\alpha M = \Gamma_\pi^2(\Gamma_\pi^\beta M)$; if α is a limit ordinal, let $\Gamma_\pi^\alpha M = \bigcap_{\beta<\alpha}\Gamma_\pi^\beta M$. It still follows that $\Gamma M \subset \Gamma_\pi^\alpha M$ for all ordinals α. But the lower central series always decreases so $\Gamma M = \Gamma_\pi^\gamma M$ for some ordinal γ. We have shown already that $\pi_n(X)/\Gamma_\pi^r\pi_n(X) \to \pi_n(Y)/\Gamma_\pi^r\pi_n(Y)$ is an isomorphism for finite r. Introducing the limit ordinals, we get an isomorphism for $r = \omega$ and the argument of Lemma 8^{bis}.25 works for the higher ordinals. Thus, $\pi_n(f)$ induces an isomorphism $\pi_n(X)/\Gamma\pi_n(X) \to \pi_n(Y)/\Gamma\pi_n(Y)$ and so, by the Five-lemma, $\pi_n(f)$ is a monomorphism. □

A characterization of nilpotent spaces

In Chapter 4 (Theorem 4.35) we constructed the Postnikov tower of a space and stated that, for simply-connected spaces, the fibrations in the tower could be taken to be **principal**, that is, each $p_n\colon P_nX \to P_{n-1}X$ is a pullback of the path-loop fibration over the Eilenberg-Mac Lane space $K(\pi_n(X), n+1)$ via a k-invariant, $k^n\colon P_{n-1}X \to K(\pi_n(X), n+1)$:

$$\begin{tikzcd}
K(\pi_n(X),n) \arrow[r, equal] \arrow[d, hook] & K(\pi_n(X),n) \arrow[d, hook] \\
P_nX \arrow[r] \arrow[d, "p_n"'] & PK(\pi_n(X),n+1) \arrow[d] \\
P_{n-1}X \arrow[r, "k^n"] & K(\pi_n(X),n+1).
\end{tikzcd}$$

We next give a proof of this property of simply-connected spaces and generalize it to nilpotent spaces.

Lemma 8^{bis}.28. *Let A be a finitely generated abelian group and let E and B be spaces of finite type. A fibration $K(A,n) \hookrightarrow E \xrightarrow{p} B$ is principal if and only if it is simple, that is, the action of $\pi_1(B)$ on $K(A,n)$ is trivial.*

PROOF: Let's assume that $p\colon E \to B$ is principal and it is pulled back over a classifying map $\theta\colon B \to K(A, n+1)$. The relevant part of the long exact sequence of homotopy groups may be written

$$\begin{tikzcd}
0 \arrow[r] & \pi_{n+1}(E) \arrow[r, "p_*"] \arrow[d, "="'] & \pi_{n+1}(B) \arrow[r, "\tau"] \arrow[d, "="'] & \pi_n(K(A,n)) \arrow[r] \arrow[d, "="'] & \pi_n(E) \arrow[r, "p_*"] \arrow[d, "="'] & \pi_n(B) \arrow[r] \arrow[d, "="'] & 0 \\
0 \arrow[r] & \pi_{n+1}(E) \arrow[r, "p_*"'] & \pi_{n+1}(B) \arrow[r, "\theta_*"'] & \pi_{n+1}(K(A,n+1)) \arrow[r] & \pi_n(E) \arrow[r] & \pi_n(B) \arrow[r] & 0.
\end{tikzcd}$$

The action of $\pi_1(B)$ on A can be identified in the second row with the action of the fundamental group of the total space of the fibration θ on the base space $K(A, n+1)$. But this factors through the action of the fundamental group of $K(A, n+1)$, which is trivial. Hence, the fibration is simple.

Suppose next that $\pi_1(B)$ acts trivially on $A = H_n(K(A,n))$. Consider the cohomology Leray-Serre spectral sequence for the fibration with coefficients in the abelian group A. Then, $E_2^{0,n} \cong H^n(K(A,n);A)$ contains the fundamental class $\imath$ corresponding to the identity map on $K(A,n)$. Since $K(A,n)$ is $(n-1)$-connected, the first differential to arise on $\imath$ is the transgression d_{n+1}, and this gives a class $d_{n+1}(\imath) = [\theta] \in H^{n+1}(B;A)$; we can form the pullback over $\theta\colon B \to K(A,n+1)$. This produces a space E_θ together with a mapping $g\colon E \to E_\theta$. Checking the long exact sequence of homotopy groups, g induces an isomorphism on homotopy, and so, in the category of spaces of the homotopy type of CW-complexes of finite type, g is a homotopy equivalence, and p is a principal fibration. □

It follows immediately from the lemma that a simply-connected space X has a Postnikov tower of principal fibrations. For an arbitrary space X, let $\{P_nX, p_n, f_n\}$ denote its Postnikov tower. We say that $p_n\colon P_nX \to P_{n-1}X$ **admits a principal refinement** if there is a sequence of principal fibrations

$$P_nX = P_{n,c}X \xrightarrow{q_c} P_{n,c-1}X \xrightarrow{q_{c-1}} \cdots \xrightarrow{q_3} P_{n,2}X \xrightarrow{q_2} P_{n,1}X = P_{n-1}X$$

with $p_n = q_2 \circ q_3 \circ \cdots \circ q_c$. With this extension of the notion of a principal fibration, we can now give a characterization of nilpotent spaces.

Theorem 8^{bis}.29. *A space X is nilpotent if and only if every stage of its Postnikov tower admits a principal refinement.*

Proof: Since each q_j is a principal fibration, we can write its classifying map as $\theta_{n,j}\colon P_{n,j}X \to K(A_{n,j}, n+1)$. We proceed by induction. By the properties of a Postnikov tower, $\pi_n(P_{n,1}X) = \pi_n(P_{n-1}X) = \{0\}$ and so $\pi_1(X)$ acts trivially (hence nilpotently) on $\pi_n(P_{n,1}X)$. Suppose that $\pi_1(X)$ acts nilpotently on $\pi_n(P_{n,j-1}X)$ of nilpotency class $\leq j-1$. View the k-invariant $\theta_{n,j}$ as a fibration (up to homotopy) and $q_j\colon P_{n,j}X \to P_{n,j-1}X$ as the inclusion of the fibre. By Proposition 8^{bis}.21, $\pi_1(X)$ acts nilpotently on $\pi_n(P_{n,j}X)$ of class $\leq j$. By induction, $\pi_1(X)$ acts nilpotently on $\pi_n(P_{n,c}X) \cong \pi_n(P_nX) \cong \pi_n(X)$.

Suppose that X is a nilpotent space and $\pi = \pi_1(B)$. The lower central series for $\pi_n(X)$ as a π-module has the form

$$\{0\} \subset \Gamma_\pi^c\pi_n(X) \subset \Gamma_\pi^{c-1}\pi_n(X) \subset \cdots \subset \Gamma_\pi^2\pi_n(X) \subset \pi_n(X).$$

By construction each quotient $\Gamma_\pi^t\pi_n(X)/\Gamma_\pi^{t+1}\pi_n(X)$ is a trivial π-module. Consider the fibration $p_n\colon P_nX \to P_{n-1}X$. The homology Leray-Serre spectral sequence (Lemma 8^{bis}.27) for this fibration gives the exact sequence

$$H_{n+1}(P_nX) \xrightarrow{p_{n*}} H_{n+1}(P_{n-1}X) \xrightarrow{d^{n+1}} \pi_n(X)/\Gamma_\pi^2\pi_n(X) \to \cdots .$$

Consider the cohomology Leray-Serre spectral sequence with coefficients in $(\pi_n X)_\pi = \pi_n(X)/\Gamma^2_\pi \pi_n(X)$ for which

$$E_2^{p,q} \cong H^p(P_{n-1}X; \mathcal{H}^q(K(\pi_n(X), n); (\pi_n(X))_\pi)).$$

There is a class $\jmath \in H^n(K(\pi_n(X), n); (\pi_n(X))_\pi)$ that represents the quotient $\pi_n(X) \to (\pi_n(X))_\pi$. This class transgresses to a class $[l_2]$ lying in $H^{n+1}(P_{n-1}X; (\pi_n(X))_\pi)$ which represents $H_{n+1}(P_{n-1}X) \xrightarrow{d^{n+1}} (\pi_n(X))_\pi$ and for which we take a representative $l_2\colon P_{n-1}X \to K((\pi_n(X))_\pi, n+1)$. Let $q_2\colon P_{n,2}X \to P_{n-1}X$ be the pullback of the path-loop fibration over l_2 and let $u_2\colon P_nX \to P_{n,2}X$ be a lifting of p_n through $P_{n,2}X$. Such a lifting exists because $l_2 \circ p_n \simeq *$.

We can modify u_2 to be a fibration and consider a portion of the homotopy exact sequences

$$\begin{tikzcd}
 & \Gamma^2_\pi \pi_n(X) \arrow[r, "\cong"] \arrow[d] & \pi_n(\text{fibre}(u_2)) \arrow[d] & & \\
0 \arrow[r] & \pi_n(X) \arrow[r] \arrow[d] & \pi_n(P_nX) \arrow[r, "p_{n*}"] \arrow[d, "u_{2*}"] & \pi_n(P_{n-1}X) \arrow[r] & \cdots \\
0 \arrow[r] & (\pi_n(X))_\pi \arrow[r, "\cong"'] & \pi_N(P_{n,2}X) \arrow[r, "l_{2*}"'] & \pi_n(P_{n-1}X) \arrow[r] & \cdots .
\end{tikzcd}$$

From this diagram we see that the fibre of u_2 is $K(\Gamma^2_\pi \pi_n(X), n)$. If we repeat this construction with u_2 replacing p_n, then we get a space $P_{n,3}X$ together with a principal fibration $q_3\colon P_{n,3}X \to P_{n,2}X$. Continuing in this way, if X is nilpotent, we eventually get to $\Gamma^{c+1}_\pi \pi_n(X) = \{0\}$ and the process stops with $u_c = q_c$ and p_n refined by principal fibrations. □

The sequence of k-invariants that a tower of principal fibrations admits may be applied to many problems in classical homotopy theory. For example, the k-invariants are the data for classical obstruction arguments. Another application was introduced by [Sullivan71] in his work on the Adams conjecture. [Serre53] showed, in his development of classes of abelian groups, that homotopy theory can become simpler when viewed one prime at a time. Making this notion topological rather than algebraic is the goal of localization at a prime. To localize a space X at a prime p, first consider the **ring of integers localized at the prime** p, denoted $\mathbb{Z}_p$, and given by the subring of $\mathbb{Q}$ of fractions a/b with b relatively prime to p. The functor on abelian groups, $A \mapsto A \otimes \mathbb{Z}_p$, is called **localization** at the prime p; it eliminates all torsion prime to p and so leaves only the p-primary data. This functor can be extended to spaces by modifying the refinement of the Postnikov tower by composing the classifying maps $\theta_{n,j}$ with the mapping induced by the localization, $K(A_{n,j}, n) \to K(A_{n,j} \otimes \mathbb{Z}_p, n)$, and then pulling back carefully. The resulting space X_p has homotopy groups $\pi_n(X_p) \cong \pi_n(X) \otimes \mathbb{Z}_p$ and integral homology groups $H_n(X_p) \cong H_n(X) \otimes \mathbb{Z}_p$.

Later in the chapter, we will present an alternate construction of the localization of a space, due to [Bousfield-Kan72] and carried out simplicially.

Convergence of the Eilenberg-Moore spectral sequence $\textcircled{\cancel{N}}$

Theorem 8^{bis}.23, the generalized Whitehead theorem, illustrates how the nilpotence condition can control the effect of the fundamental group. The relations between the homotopy groups of a space and their nilpotent completions provide the data for measuring the departure from the simply-connected case of the Whitehead theorem. Another naive situation in which simple connectivity plays a role is the convergence of the Eilenberg-Moore spectral sequence. The goal of this section is to prove the following result of [Dwyer74] that shows how the nilpotence of a certain action of the fundamental group is decisive in generalizing the naive convergence criterion.

Theorem 8^{bis}.30. *Suppose $F \hookrightarrow E \xrightarrow{p} B$ is a fibration with all spaces connected, and A is an abelian group. Then the Eilenberg-Moore spectral sequence for the fibre of p converges strongly to $H_*(F;A)$ if and only if $\pi_1(B)$ acts nilpotently on $H_i(F;A)$ for all $i \geq 0$.*

Following [Rector71] (§8.3) we associate to the pullback data $X \xrightarrow{f} B \xleftarrow{p} E$ the cosimplicial space (the geometric cobar construction) $G^\bullet(X,B,E)$ where $G^n(X,B,E) = X \times B^{\times n} \times E$ for $n \geq 0$ and with coface and codegeneracy maps given by

$$
\begin{aligned}
d^i(x, b_1, \ldots, b_n, e) &= \begin{cases} (x, f(x), b_1, \ldots, b_n, e) & i = 0, \\ (x, b_1, \ldots, b_i, b_i, \ldots, b_n, e) & 1 \leq i \leq n, \\ (x, b_1, \ldots, b_n, p(e), e) & i = n+1. \end{cases} \\
s^j(x, b_1, \ldots, b_n, e) &= (x, b_1, \ldots, \widehat{b_{i+1}}, \ldots, b_n, e), \qquad 0 \leq i \leq n-1.
\end{aligned}
$$

In this discussion we take all spaces involved to be simplicial sets. Thus $G^\bullet(X,B,E)$ is a cosimplicial simplicial set. We explore the combinatorial structure of such an object in what follows.

Let A denote an abelian group and X, a simplicial set (§4.2). Then we define the simplicial abelian group $A \otimes X$ by $(A \otimes X)_n = \bigoplus_{x \in X_n} A$, for $n \geq 0$, with face and degeneracy maps induced by the maps on the generators and extended to be A-linear. It follows that $\pi_*(A \otimes X) \cong H_*(X;A)$ and problems concerning homology become open to homotopy methods.

In homological algebra, the basic datum of a resolution of a module M is the augmentation $F_\bullet \xrightarrow{\varepsilon} M \to 0$. We can view a cosimplicial space $Y^\bullet$ as a kind of resolution (for example, when constructed from a triple; [Bousfield-Kan72, I,§5]). We consider all possible augmentations of $Y^\bullet$, that is, maps $\epsilon\colon Z \to Y^0$ satisfying $d^0 \circ \epsilon = d^1 \circ \epsilon$. The **maximal augmentation** associated to $Y^\bullet$ is the subspace $aY^\bullet$ of Y^0 that gives the equalizer (as simplicial sets) of the coface mappings $d^0, d^1\colon Y^0 \to Y^1$. In detail, the space $aY^\bullet$ is given by $aY^\bullet = \{y \in Y^0 \mid d^0(y) = d^1(y)\}$. The maximal augmentation has the following characterization in the category of cosimplicial spaces.

Lemma 8^{bis}.31. *The maximal augmentation $aY^\bullet$ of a cosimplicial space $Y^\bullet$ is the simplicial set* **CoSimp**$(*, Y^\bullet)$ *of cosimplicial maps from the constant cosimplicial space $*$.*

We leave the proof of the lemma to the reader. The Hom-set of cosimplicial maps between $X^\bullet$ and $Y^\bullet$, **CoSimp**$(X^\bullet, Y^\bullet)$, has the structure of a simplicial set with n-simplices given by the cosimplicial maps $\Delta[n] \times X^\bullet \to Y^\bullet$. Here $\Delta[n]_\bullet$ denotes the standard simplicial n-simplex, whose s-simplices are given by

$$\Delta[n]_s = \{\langle x_0, x_1, \ldots, x_s\rangle \mid 0 \le x_0 \le x_1 \le \cdots \le x_s \le n\}.$$

The face and degeneracy maps on **CoSimp**$(X^\bullet, Y^\bullet)$ are induced by the standard maps. The inclusions $\varepsilon_i\colon \Delta[n] \to \Delta[n+1]$ are given by $\varepsilon_i(\langle x_0, x_1, \ldots, x_s\rangle) = \langle X_0, X_1, \ldots, X_s\rangle$, where $X_j = x_j$, if $j < i$, and $X_j = x_j + 1$, if $j \ge i$. The face mapping is given by $d_i\colon \Delta[n] \times X^\bullet \xrightarrow{\varepsilon_i \times 1} \Delta[n+1] \times X^\bullet \to Y^\bullet$. The degeneracy maps are defined by the combinatorial collapse onto the j^{th} face, namely $\eta_j\colon \Delta[n] \to \Delta[n-1]$, given by $\eta_j(\langle x_0, x_1, \ldots, x_s\rangle) = \langle X_0, X_1, \ldots, X_s\rangle$, where $X_l = x_l$, if $l < j$, and $X_l = x_l - 1$, if $l \ge j$. Thus $s_j\colon \Delta[n] \times X^\bullet \xrightarrow{\eta_j \times 1} \Delta[n-1] \times X^\bullet \to Y^\bullet$.

A desirable property of resolutions is homotopy invariance. For cosimplicial spaces, we want a similar property—if $f\colon Y^\bullet \to Z^\bullet$ is a morphism of cosimplicial spaces that satisfies the condition that $f\colon Y^n \to Z^n$ is a homotopy equivalence of simplicial sets for all n, then f ought to induce a homotopy equivalence of maximal augmentations. However, this is too much to ask for. The fix for this desideratum is to replace the construction of the maximal augmentation with one that is more robust homotopically.

Definition 8^{bis}.32. *Given a cosimplicial space $Y^\bullet$, let $\mathrm{Tot}(Y^\bullet)$ denote the simplicial set* **CoSimp**$(\Delta^\bullet, Y^\bullet)$ *where $\Delta^\bullet$ denotes the cosimplicial space with $\Delta[n]$ at level n and coface and codegeneracy mappings induced by the canonical face inclusions, ε_i, and projections, η_j, respectively.*

This functor was introduced by [Bousfield-Kan72] and forms the basis for their study of localization and completion. $\mathrm{Tot}(Y^\bullet)$ can be built up canonically from a tower of fibrations. Let $\Delta^{\bullet(s)}$ denote the s-skeleton of the cosimplicial space $\Delta^\bullet$, that is, at level n, one takes the s-skeleton of $\Delta[n]$. Define

$\mathrm{Tot}_s(Y^\bullet) = \mathbf{CoSimp}(\Delta^{\bullet(s)}, Y^\bullet)$. The cofibrations $\Delta^{\bullet(s)} \to \Delta^{\bullet(s+1)}$ induce fibrations $\mathrm{Tot}_{s+1}(Y^\bullet) \to \mathrm{Tot}_s(Y^\bullet)$, whose inverse limit is $\mathrm{Tot}_\infty(Y^\bullet) = \mathrm{Tot}(Y^\bullet)$. Notice that $\mathrm{Tot}_0(Y^\bullet) = Y^0$, and if $\epsilon\colon Z \to Y^\bullet$ is any augmentation, then ϵ induces a mapping $Z \to \mathrm{Tot}_s(Y^\bullet)$, for all $s \leq \infty$.

A tower of fibrations gives rise to an exact couple based on the long exact sequences of homotopy groups. The E^1-term is determined by the homotopy groups of the fibres of $\mathrm{Tot}_s \to \mathrm{Tot}_{s-1}$. A typical fibre takes the form $\Omega^s((NY^\bullet)^s)$ where $(NY^\bullet)^s$ may be written as $Y^s \cap \ker s^0 \cap \cdots \cap \ker s^{s-1}$ when the simplicial sets at each level of $Y^\bullet$ are **fibrant** (that is, $Y^n \to *$ is a fibration for all n). It follows from the grading for the exact couple that this is a second quadrant spectral sequence. There are general conditions for its strong convergence to $\pi_*(\mathrm{Tot}(Y^\bullet))$ (see [Bousfield-Kan72, IX, §5]). We will obtain the Eilenberg-Moore spectral sequence in this manner by taking the homotopy spectral sequence associated to the tower of fibrations $\{\mathrm{Tot}_s(A \otimes G^\bullet(X, B, E))\}$.

In the category of cosimplicial spaces we find the usual notions of homotopy theory such as fibrations, cofibrations, and homotopy equivalences. The case of interest is the following diagram depicting a fibration of cosimplicial spaces along with an augmenting fibration of spaces:

$$\begin{tikzcd}
F \arrow[r, "\epsilon"] \arrow[d, hook] & G^\bullet(*, B, E) \arrow[d, hook] \\
E \arrow[r, "\epsilon"] \arrow[d, "p"'] & G^\bullet(B, B, E) \arrow[d, "q"] \\
B \arrow[r, "="] & B.
\end{tikzcd}$$

Here B denotes the constant cosimplicial space with B at all levels and the identity map for all coface and codegeneracy maps. The maps for $G^\bullet(B, B, E)$ are given by $\mathrm{id}\colon B \to B \leftarrow E\colon p$. The mapping q is given by first projection off the product $B \times B^{\times n} \times E$. Thus, at each level, we have a trivial fibration and so $\pi_1(B)$ acts trivially on each fibre $G^n(*, B, E)$. We next show that the action of $\pi_1(B)$ on $H_i(F; A)$ is compatible via the augmentation with this trivial action.

Proposition 8^{bis}.33. *The augmentation map $\epsilon\colon F \to * \times E = G^0(*, B, E)$ induces a $\pi_1(B)$-equivariant homomorphism $\epsilon_*\colon H_*(F; A) \to H_*(E; A)$, where $\pi_1(B)$ acts trivially on $H_*(E; A)$*

PROOF: We argue with spaces and lifting functions as in §4.3. The simplicial versions of these structures can be found in [May67]. The pullback spaces for the fibrations p and q are given by $\Omega_p = \{(\lambda, e) \mid \lambda \in WB, e \in E, \lambda(0) = p(e)\}$, and $\Omega_q = \{(\lambda, b, e) \mid \lambda \in WB, (b, e) \in B \times E, \lambda(0) = b\}$. The augmentation maps induce a mapping between these pullbacks $\Omega_p \to \Omega_q$, given explicitly by $(\lambda, e) \mapsto (\lambda, p(e), e)$. This gives rise to the diagram

$$\begin{tikzcd}
\Omega B \times F \arrow[r, hook] \arrow[d] & \Omega_p \arrow[r, "\Lambda"] \arrow[d] & WE \arrow[r, "\mathrm{ev}_1"] \arrow[d] & F \arrow[d] \\
\Omega B \times (* \times E) \arrow[r, hook] & \Omega_q \arrow[r, "\Lambda'"'] & W(B \times E) \arrow[r, "\mathrm{ev}_1"'] & (* \times E).
\end{tikzcd}$$

As described in §4.3, the lifting function for the trivial fibration is given by $\Lambda'(\lambda, b, e) = (\lambda, c_e)$, where c_e denotes the constant path at e. Since the action of $\pi_1(B)$ is induced by these composites, compatibility of the actions is equivalent to the homotopy commutativity of this diagram. Let $H\colon \Omega B \times F \times I \to (* \times E)$ be given by $H((\omega, y), t) = (*, \Lambda(\omega, y)(t))$. Then H makes the leftmost square commute up to homotopy and so proves the proposition. □

The fibration of cosimplicial spaces $G^\bullet(*, B, E) \to G^\bullet(B, B, E) \to B$ provides control of the $\pi_1(B)$-action in the tower of fibrations that give rise to the Eilenberg-Moore spectral sequence.

Lemma 8^{bis}.34. *For all* $i \geq 1$, $\pi_i(\mathrm{Tot}_s(A \otimes G^\bullet(*, B, E)))$ *is a nilpotent* $\pi_1(B)$-*module.*

PROOF: We prove this by induction over s. When $s = 0$, we have the trivial fibration $E \to B \times E \to B$ that describes the 0-level of the fibration of cosimplicial spaces. Thus $\pi_1(B)$ acts trivially on $\pi_i(\mathrm{Tot}_0(A \otimes G^\bullet(*, B, E)))$.

By induction we consider the fibration

$$\mathrm{Tot}_n(A \otimes G^\bullet(*, B, E)) \to \mathrm{Tot}_{n-1}(A \otimes G^\bullet(*, B, E)).$$

[Bousfield-Kan72, X, §6] give an explicit expression for the fibre of this fibration from which we deduce its structure as a $\pi_1(B)$-module. To wit, the fibre of $\mathrm{Tot}_n(Y^\bullet) \to \mathrm{Tot}_{n-1}(Y^\bullet)$, for any cosimplicial space $Y^\bullet$, is given by the function space $\mathrm{Hom}((S^n, *), (NY^n, *))$ where $NY^n = \ker(Y^n \overset{s}{\to} M^{n-1}Y^\bullet)$ and $M^{n-1}Y^\bullet$ is the $(n-1)^{st}$ *matching space* consisting of simplices in $(Y^{n-1})^{\times n}$, written $(x^0, \ldots, x^{n-1})$, that satisfy $s^i x^j = s^{j-1} x^i$ whenever $0 \leq i < j \leq n-1$. The mapping $s\colon Y^n \to M^{n-1}Y^\bullet$ is given by $y \mapsto (s^0 y, \ldots, s^{n-1} y)$. In the case of a cosimplicial simplicial abelian group, the homotopy groups of the fibre may be written

$$\begin{aligned}\pi_i(\text{fibre}(\mathrm{Tot}_n(Y^\bullet) \to \mathrm{Tot}_{n-1}(Y^\bullet))) &\cong \pi_{i+n}(NY^n) \\ &= \pi_{i+n}(Y^n) \cap \ker s^0 \cap \cdots \cap \ker s^{n-1}.\end{aligned}$$

When $Y^\bullet = A \otimes G^\bullet(*, B, E)$, the homotopy groups of the fibre have a $\pi_1(B)$-action inherited from the inclusion into $A \otimes G^n(*, B, E)$. However,

$\pi_{i+n}(A\otimes G^n(*,B,E))$ is a trivial $\pi_1(B)$-module, and so then are the homotopy groups of the fibre of $\mathrm{Tot}_n(Y^\bullet)\to\mathrm{Tot}_{n-1}(Y^\bullet)$. By induction, we assume that the groups $\pi_j(\mathrm{Tot}_{n-1}(A\otimes G(*,B,E)))$ are nilpotent $\pi_1(B)$-modules. The long exact sequence of homotopy groups for the fibration $\mathrm{Tot}_n(Y^\bullet)\to\mathrm{Tot}_{n-1}(Y^\bullet)$ and the triviality of the $\pi_1(B)$-action on the homotopy groups of the fibre complete the induction. □

From the lemma we can deduce half of the proof of Theorem 8^{bis}.30. Suppose that the spectral sequence converges strongly to $H_*(F;A)$. Then there is a filtration of $H_i(F;A)$ for each i with $E^\infty_{p,i-p}$ isomorphic to the associated graded group to this filtration. Strong convergence implies that the direct limit of the sequence

$$\cdots\to\pi_i(A\otimes\mathrm{Tot}_s(A\otimes G^\bullet(*,B,E)))\to\pi_i(\mathrm{Tot}_{s-1}(A\otimes G^\bullet(*,B,E)))\to\cdots$$

vanishes and so there is an injection

$$\begin{aligned}E^\infty_{p,*}\to R_p=\bigcap\nolimits_r \mathrm{im}(\pi_*(\mathrm{Tot}_{p+r}(A\otimes G^\bullet)&\to\mathrm{Tot}_p(A\otimes G^\bullet))\\ &\subset\pi_*(\mathrm{Tot}_p(A\otimes G^\bullet(*,B,E))).\end{aligned}$$

It follows that each $E^\infty_{p,*}$ is a nilpotent $\pi_1(B)$-module. Strong convergence also implies that the nonzero $\pi_1(B)$-modules $E^\infty_{p,i-p}$ are finite in number. Arguing inductively using Corollary 8^{bis}.19 we have proved that $H_i(F;A)$ is a nilpotent $\pi_1(B)$-module.

To prove the other half of Theorem 8^{bis}.30 we use the towers of fibrations that arise from the application of the functor Tot_s to the cosimplicial fibration $G^\bullet(*,B,E)\to G^\bullet(B,B,E)\to B$. The augmentation from the fibration p may be depicted in the diagram:

$$\begin{tikzcd}
F \arrow[r, "\epsilon"] \arrow[d] & \mathrm{Tot}_0\, G^\bullet(*,B,E) \arrow[d] & \mathrm{Tot}_1\, G^\bullet(*,B,E) \arrow[l] \arrow[d] & \cdots \arrow[l] & \mathrm{Tot}\, G^\bullet(*,B,E) \arrow[l] \arrow[d] \\
E \arrow[r, "\epsilon"] \arrow[d] & \mathrm{Tot}_0\, G^\bullet(B,B,E) \arrow[d] & \mathrm{Tot}_1\, G^\bullet(B,B,E) \arrow[l] \arrow[d] & \cdots \arrow[l] & \mathrm{Tot}\, G^\bullet(B,B,E) \arrow[l] \arrow[d] \\
B \arrow[r, "="'] & B & B \arrow[l, "="'] & \cdots \arrow[l, "="'] & B. \arrow[l, "="']
\end{tikzcd}$$

Lemma 8^{bis}.35. $\mathrm{Tot}(G^\bullet(B,B,E))\simeq E$.

PROOF: The projection off the last coordinate $G^n(B,B,E)\to E$ provides an inverse to the augmentation $E\to G^\bullet(B,B,E)$. □

It follows that we can compare the augmentation fibration with the limit fibration. The nilpotency condition plays a role in the following proposition that is a form of the Zeeman comparison theorem. The proposition was known in the early 1970's—it is stated explicitly by [Hilton-Roitberg76].

Proposition 8^{bis}.36. *Suppose $F \hookrightarrow E \to B$ and $F' \hookrightarrow E' \to B$ are fibrations with B connected, and $f\colon E \to E'$ is a map over B inducing an isomorphism on homology. If $\pi_1(B)$ acts nilpotently on $H_i(F)$ and on $H_i(F')$, for all i, then $f|\colon F \to F'$ induces an isomorphism on homology.*

PROOF: We proceed by induction on the degree i of $H_i(f|)$. In the case $i = 0$, $H_0(f) = H_0(f|)$ because B is connected.

Suppose $H_i(f|)$ is an isomorphism for $0 \leq i \leq n-1$. This implies that the E^2-terms of the associated Leray-Serre spectral sequences are isomorphic in bidegrees $(*, i)$ for $i \leq n-1$. We consider the morphism of spectral sequences in bidegrees $(0, n)$ and $(1, n)$, where we have $E^2_{0,n} \cong H_0(B, \mathcal{H}_n(F)) \to H_0(B, \mathcal{H}_n(F')) \cong {E'}^2_{0,n}$. By Proposition 8^{bis}.4,

$$H_0(B, \mathcal{H}_n(F)) \cong H_0(\pi, H_n(F)) = (H_n(F))_\pi,$$

where $\pi = \pi_1(B)$. By Theorem 8^{bis}.11, $E^2_{1,n} \cong H_1(\pi, H_n(F))$. On the vertical edge of the spectral sequence the map of spectral sequences gives

$$\begin{tikzcd}
E^2_{0,n} \arrow[r, two heads] \arrow[d] & E^3_{0,n} \arrow[r, two heads] \arrow[d] & \cdots \arrow[r, two heads] & E^{n+1}_{0,n} = E^\infty_{0,n} \arrow[d, "\cong"'] \\
{E'}^2_{0,n} \arrow[r, two heads] & {E'}^3_{0,n} \arrow[r, two heads] & \cdots \arrow[r, two heads] & {E'}^{n+1}_{0,n} = {E'}^\infty_{0,n}.
\end{tikzcd}$$

Since the E^2-terms are isomorphic in bidegrees $(*, i)$ for $i \leq n-1$, the differentials arising to make the successive epimorphisms along the vertical edges are the same in each spectral sequence and so we conclude that $H_0(\pi, H_n(F))$ is isomorphic to $H_0(\pi, H_n(F'))$ via $H_n(f|)$. Similarly, we find that $H_1(\pi, H_n(F))$ maps onto $H_1(\pi, H_n(F'))$ via $H_n(f|)$. Theorem 8^{bis}.16 implies that $H_n(F)$ is isomorphic to $H_n(F')$, and the inductive step follows. $\square$

The second half of the proof of the Theorem 8^{bis}.30 follows because the homotopy spectral sequence for the tower of fibration $\{\mathrm{Tot}_s(A \otimes G^\bullet(*, B, E))\}$ converges to $\pi_*(\mathrm{Tot}(A \otimes G^\bullet(*, B, E)))$. Proposition 8^{bis}.36 implies that $\pi_*(\mathrm{Tot}(A \otimes G^\bullet(*, B, E))) \cong H_*(F; A)$.

Theorem 8^{bis}.30 has been extended to connective generalized homology theories ([Bousfield87]), nonconnected bases B ([Dror-Farjoun–Smith, J90], a useful case when dealing with function spaces) and to pullback fibre squares with data $X \xrightarrow{f} B \xleftarrow{p} Y$ for which the set $\pi_0(X) \times_{\pi_0(B)} \pi_0(Y)$ is finite and, for all $y \in Y$, $\pi_1(B, p(y))$ acts nilpotently on $H_*((Y)_y)$ where $(Y)_y$ denotes the component of Y containing y ([Shipley96]).

The development of convergence criteria for the Eilenberg-Moore spectral sequence is, in fact, a spinoff of the investigation of the general convergence properties of the **Bousfield-Kan spectral sequence**.

Theorem 8^{bis}.37. *Given a fibrant, pointed, cosimplicial space $Y^\bullet$, there is a spectral sequence associated to the tower of fibrations $\{\mathrm{Tot}_n(Y^\bullet) \to \mathrm{Tot}_{n-1}(Y^\bullet)\}$ with*

$$E_1^{s,t}(Y^\bullet) \cong \pi_t(Y^s) \cap \ker s^0 \cap \cdots \cap \ker s^{s-1}, \qquad t \geq s \geq 0$$

and converging under favorable conditions to $\pi_*(\mathrm{Tot}(Y^\bullet))$.

General results indicating favorable conditions were obtained by [Bousfield87], [Shipley96], and [Goodwillie98]. The fundamental example introduced by [Bousfield-Kan72] is the cosimplicial space associated to the completion of a space with respect to a ring R.

The R-**completion** of a pointed space (X, x_0) is obtained by applying the totalization functor, Tot, to the cosimplicial space $R^\bullet X$ obtained from the triple $\{R, \phi, \psi\}$ as follows: If (X, x_0) is a pointed simplicial set, then define the simplicial R-module RX by $(RX)_n = R \otimes X_n / R \otimes x_0$. The natural transformation $\phi_X \colon X \to RX$ is defined by $x \mapsto [1 \otimes x]$, and the natural transformation $\psi_X \colon R^2X \to RX$ is given by $[r \otimes [s \otimes x]] \mapsto [rs \otimes x]$. The R-completion of X is defined by

$$R_\infty X = \mathrm{Tot}(R^\bullet X) = \mathrm{Hom}(\Delta^\bullet, R^\bullet X),$$

where $R^kX = R(R^{k-1}X)$ and $R^0X = RX$. The cosimplicial structure is based on the natural transformations ϕ and ψ, with the coface and codegeneracy maps given by

$$\begin{aligned} d^i &\colon R^kX \to R^{k+1}X, \quad d^i = R^i(\phi_{R^{k-i}X}), \\ s^j &\colon R^kX \to R^{k-1}X, \quad s^j = R^j(\psi_{R^{k-j}X}). \end{aligned}$$

It follows from the properties of Tot that $R_\infty X$ is the inverse limit of a tower of fibrations $R_sX \to R_{s-1}X$ where $R_sX = \mathrm{Tot}_s(R^\bullet X)$. This tower of fibrations is augmented by a family of mappings $f_s \colon X \to R_sX$ and it leads to the spectral sequence of [Bousfield-Kan72].

When R is a subring of $\mathbb{Q}$, then, one can prove that, for some set P of primes,

$$R = \mathbb{Z}_P = \{a/b \in \mathbb{Q} \mid p \nmid b, \text{ for all } p \in P\}.$$

The R-completion of a nilpotent space (X, x_0) coincides in this case with its $\mathbb{Z}_P$-localization ([Bousfield-Kan72, V, §4]). Thus (co)simplicial techniques generalize the localization construction via Postnikov towers of [Sullivan71] to general rings. The basic algebraic condition on the ring R that guarantees good completion properties is that R be **solid**, that is, the multiplication on R induces an isomorphism $R \otimes R \to R$.

When $f_{\infty *} \colon \tilde{H}_*(X; R) \to \tilde{H}_*(R_\infty X; R)$ is an isomorphism, then we say that X is R-**good**. For R-good spaces the R-completion, $f_\infty \colon X \to$

$R_\infty X$, enjoys certain universal properties. For example, a mapping $f\colon X \to Y$ induces an isomorphism $H_*(f)\colon H_*(X;R) \to H_*(Y;R)$ if and only if $R_\infty f\colon R_\infty X \to R_\infty Y$ is a homotopy equivalence ([Bousfield-Kan72, I.5.5]). However, there are spaces that are not R-good—for example, an infinite wedge of circles is not $\mathbb{Z}$-good. Nilpotent spaces are $\mathbb{Z}$-good. With this language we can describe the solution to the natural question—what is the target of the Eilenberg-Moore spectral sequence in general? [Dwyer75] found the answer for the Eilenberg-Moore spectral sequence associated to the fibre of a fibration $p\colon E \to B$: The spectral sequence converges to the homology of the **nilpotent completion** of the fibration, that is, to $H_*(\tilde{F};R)$, where $\tilde{F}$ is the fibre of the fibration $R_\infty p\colon R_\infty E \to R_\infty B$.

Completion and localization constructions have become fundamental in homotopy theory and a complete exposition of these ideas would take us too far afield. Nice expositions of this circle of ideas may be found in [Sullivan71], [Mimura-Nishida-Toda71], [Hilton75], [Hilton-Mislin-Roitberg75], and [Arkowitz76]. The most complete exposition of these ideas is the work of [Bousfield-Kan72].

A consequence of the cosimplicial construction of the R-completion is a result of [Dror73] that shows the extent to which nilpotent spaces approximate general homotopy types. To state precisely what sort of approximation we mean, we compare a connected space X with the associated tower of fibrations $\{R_sX\}$. By the definition of Tot_s, we have the augmentation mappings $f_s\colon X \to R_sX$ for all $s \geq 0$ and these mappings are compatible with the sequence of fibrations $R_{s+1}X \to R_sX$. Thus the mappings $\{f_s\}$ determine a mapping of towers of spaces $\{X\} \to \{R_sX\}$.

A tower of groups $\{G_s\}$ is a sequence of homomorphisms $G_{s+1} \to G_s$ for $s \geq 0$. A homomorphism of towers of groups, $\xi\colon \{G_s\} \to \{H_s\}$, is a sequence of group homomorphisms $\xi_s\colon G_s \to H_s$, compatible with the tower mappings. The natural maps $f_s\colon X \to R_sX$ determine, for each $i \geq 0$, a homomorphism of towers of groups $f_*\colon \{H_i(X;R)\} \to \{H_i(R_sX;R)\}$.

Definition 8^{bis}.38. *A homomorphism of towers of groups, $\xi\colon \{G_s\} \to \{H_s\}$, is a* **pro-isomorphism** *if, for any group A, ξ induces an isomorphism*

$$\xi^*\colon \varinjlim \mathrm{Hom}_{\mathbf{Grp}}(H_s, A) \to \varinjlim \mathrm{Hom}_{\mathbf{Grp}}(G_s, A).$$

We leave it as an exercise to show that $\xi\colon \{G_s\} \to \{H_s\}$ is a pro-isomorphism if and only if, for each $t \geq 0$, there is a value $t' \geq t$ and a homomorphism $u_t\colon H_{t'} \to G_t$ such that the following diagram commutes:

$$\begin{tikzcd}
G_{t'} \arrow[r, "\xi_{t'}"] \arrow[d, "p^G_{t',t}"'] & H_{t'} \arrow[d, "p^H_{t',t}"] \arrow[dl, "u_t"'] \\
G_t \arrow[r, "\xi_t"'] & H_t.
\end{tikzcd}$$

Here $p^G_{t',t}$ denotes the composition $G_{t'} \to G_{t'-1} \to \cdots G_{t+1} \to G_t$ and likewise for $p^H_{t',t}$. We also leave it to the reader to show that a pro-isomorphism induces an isomorphism of limits:

$$\xi\colon \lim_{\leftarrow s} G_s \cong \lim_{\leftarrow s} H_s \text{ and } \xi\colon \lim_{\leftarrow s}{}^1 G_s \cong \lim_{\leftarrow s}{}^1 H_s.$$

We say that a pointed space (X, x_0) is R-**nilpotent** if X is nilpotent and, for each $n \geq 1$, there is a central series of $\pi_1(X, x_0)$-modules

$$\pi_n(X, x_0) = M_1 \supset M_2 \supset \cdots \supset M_{c-1} \supset M_c \supset M_{c+1} = \{0\},$$

for which each subquotient M_j/M_{j+1} is a trivial $\pi_1(X, x_0)$-module and an R-module. A space is nilpotent when it is $\mathbb{Z}$-nilpotent.

Proposition 8bis.39. *For an arbitrary connected, pointed space (X, x_0), the spaces $R_sX = \operatorname{Tot}_s(R^\bullet X)$ are R-nilpotent for all $s \geq 0$. Furthermore, the natural maps $f_s\colon X \to R_sX$ induce, for all $i \geq 1$, a pro-isomorphism of towers of homology groups $f_*\colon \{H_i(X; R)\} \to \{H_i(R_sX; R)\}$.*

SKETCH OF A PROOF: The space RX is R-nilpotent since it is an H-space and an R-module. According to [Bousfield-Kan72, III.5.5], if $p\colon E \to B$ is a principal fibration with connected fibre F and any two of E, B, and F are R-nilpotent, then so is the third. Their Proposition II.2.5 asserts that $R_sX \to R_{s-1}X$ is a principal fibration with fibre a connected simplicial R-module. Thus the spaces R_sX are R-nilpotent for all $s \geq 0$.

To establish that we have a homology pro-isomorphism, we observe that $H_k(X; R) \cong \pi_k(RX, x_0)$ and so we can compare the tower of homotopy groups $\{\pi_k(RX, x_0)\}$ with $\{\pi_k(RR_sX, x_0)\}$. When comparing the spaces RX and RR_sX, we have a triple structure available and hence mappings

$$\phi\colon RX \leftrightarrow RR_sX\colon \psi \qquad \text{with } \psi\phi = \operatorname{id}.$$

[Dror73] interpolates a condition that implies that a pro-isomorphism on homotopy is induced by $\{RX\} \to \{RR_sX\}$, namely, that the map of towers $\{R_nX\} \to \{R_sR_nX\}$ induce a pro-isomorphism. He then uses the convergence of the homotopy spectral sequence associated to the tower $\{R_sX\}$ to obtain the pro-isomorphism $\{H_k(X; R)\} \to \{H_k(R_sX; R)\}$ for $k \geq 1$. □

It follows from the proposition that every connected, pointed homotopy type may represented by a tower of R-nilpotent spaces, up to homology equivalence. This approximation is analogous to the Stone-Weierstrass theorem: Every homotopy type (continuous function) may be represented by a tower of R-nilpotent spaces (a sequence of polynomials) such that $\tilde{H}_*(X; R) \cong \lim_{\leftarrow s} \tilde{H}_*(R_sX; R)$ (limits agree). To study whether a space is R-good, we can focus on the relation between $\lim_{\leftarrow s} \tilde{H}_*(R_sX; R)$ and $\tilde{H}_*(R_\infty X; R)$.

Exercises

8^{bis}.1. Show that $\pi_1(\mathbb{R}P^{2n})$ acts nonnilpotently on $\pi_{2n}(\mathbb{R}P^{2n})$.

8^{bis}.2. Show that the action $\nu_n^{E,F}\colon \pi_1(E,e)\times\pi_n(F,e)\to\pi_n(F,e)$ is well-defined and that $i_*\colon \pi_n(F,e)\to\pi_n(E,e)$ is a $\pi_1(E,e)$-module homomorphism.

8^{bis}.3. Suppose that M is a module over a group π. Show that the coinvariants M_π is the largest quotient of M on which π acts trivially. Show directly that the functor $M\mapsto M_\pi$ is right exact.

8^{bis}.4. Let π denote the cyclic group of order m, with generator $t\in\pi$. Show that the complex

$$\cdots\to W_n\to\cdots\to W_2\xrightarrow{N}W_1\xrightarrow{T}W_0\xrightarrow{\varepsilon}\mathbb{Z}\to 0$$

is a resolution of $\mathbb{Z}$ over $\mathbb{Z}\pi$, where W_k the free $\mathbb{Z}\pi$-module on a single generator w_k and boundary homomorphisms $T\colon W_{2n+1}\to W_{2n}$ given by $T(w_{2n+1}) = tw_{2n}-w_{2n}$ and $N\colon W_{2n}\to W_{2n-1}$ given by $N(w_{2n}) = w_{2n-1}+tw_{2n-1}+\cdots+t^{m-1}w_{2n-1}$.

8^{bis}.5. Suppose that π is a finitely generated group. Show that $H_i(\pi,M)$ is finitely generated whenever M is finitely generated over $\mathbb{Z}\pi$ and $i\geq 0$.

8^{bis}.6. Prove directly that $H_1(\pi)\cong\pi/[\pi,\pi]$.

8^{bis}.7. Prove Theorem 8^{bis}.11.

8^{bis}.8. Suppose that $1\to H\to\pi\to Q\to 1$ is an extension of groups. Complete the proof of Theorem 8^{bis}.14 by showing that the Q-coinvariants of the conjugation action on $H/[H,H]$ are given by $H/[\pi,H]$.

8^{bis}.9. Suppose that $1\to R\to F\to\pi\to 1$ is a presentation of the fundamental group $\pi=\pi_1(X)$ of a space X, where F and R are free groups. Prove the classic result of Hopf that $H_2(X)/h_*(\pi_2(X))\cong R\cap[F,F]/[F,R]$ where $h_*\colon\pi_2(X)\to H_2(X)$ denotes the Hurewicz homomorphism.

8^{bis}.10. Suppose that $1\to H\to\pi\to Q\to 1$ is a central extension, that is, H maps to a subgroup of the center of π. Show that there is an exact sequence:

$$H_2(\pi)\to H_2(Q)\to H\to H_1(\pi)\to H_1(Q)\to 0.$$

8^{bis}.11. Prove Corollary 8^{bis}.19.

8^{bis}.12. Suppose that $F\hookrightarrow E\xrightarrow{p}B$ is a fibration of connected spaces. Suppose that E is nilpotent. Show that $\pi_1(F)$ acts nilpotently on itself by conjugation.

8^{bis}.13. Suppose π acts nilpotently on M and $H_0(\pi,M)=\{0\}$. Conclude that $M=\{0\}$.

8$^{\text{bis}}$.14. The functor Γ associates to a π-module the submodule ΓM generated by the union of the family of all perfect submodules of M, that is, submodules N with $N = \Gamma_\pi^2 N$. Show that ΓM is also perfect and that it is the maximal π-perfect submodule of M. Show that $\Gamma M \subset \Gamma_\pi^n M$ for all n.

8$^{\text{bis}}$.15. Show that $\Gamma_\pi^n M = \Gamma_\pi^{n+1} M$ implies that $\Gamma_\pi^n M = \Gamma_\pi^{n+k} M$ for all $k \geq 0$. Thus the lower central series is a sequence of proper inclusions until it stablilizes.

8$^{\text{bis}}$.16. Show that all nilpotent spaces are π-complete.

8$^{\text{bis}}$.17. Show that the maximal augmentation of a cosimplicial space $Y^\bullet$ is given by $aY^\bullet = \mathbf{CoSimp}(*, Y^\bullet)$.

8$^{\text{bis}}$.18. If $R \subset \mathbb{Q}$ is a subring of $\mathbb{Q}$, then show that there is a set of primes P (possibly empty) for which $R = \mathbb{Z}_P$.

8$^{\text{bis}}$.19. Show that a homomorphism of towers of groups, $\xi\colon \{G_s\} \to \{H_s\}$, is a pro-isomorphism if and only if, for each $t \geq 0$, there is a value $t' \geq t$ and a homomorphism $u_t\colon H_{t'} \to G_t$ such that the following diagram commutes:

$$\begin{tikzcd}
G_{t'} \arrow[r, "\xi_{t'}"] \arrow[d, "p^G_{t',t}"'] & H_{t'} \arrow[dl, "u_t"'] \arrow[d, "p^H_{t',t}"] \\
G_t \arrow[r, "\xi_t"'] & H_t.
\end{tikzcd}$$

Show further that a pro-isomorphism induces an isomorphism of limits:

$$\xi\colon \lim_{\leftarrow s} G_s \cong \lim_{\leftarrow s} H_s \text{ and } \xi\colon \lim_{\leftarrow s}{}^1 G_s \cong \lim_{\leftarrow s}{}^1 H_s.$$