Trigonometries

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1. INTRODUCTION. Non-Euclidean (or hyperbolic) geometry was founded independently and almost simultaneously by N.I. Lobachevsky (1792–1850), J. Bolyai (1802–1860), and C.-F. Gauss (1777–1855). The manner of its discovery and development is one of the grand stories in the history of mathematics; see the accounts of [10], [13], or [16] for details. When you venture into the original documents [3], [6], it is the trigonometry of the non-Euclidean plane that is one of the principal goals of these classic works. The desiderata in these original papers were the analogues of results in spherical trigonometry, especially as presented by L. Euler (1707–1783), whose papers [7], [8], set the standard for this subject.

The basic stuff of trigonometry is the set of relations among the geometric quantities associated to triangles. On the sphere, the plane tangent to the sphere at a point may be used to discover the basic relations (this is Euler's approach [9]). In the non-Euclidean plane, the arguments for the analogous relations are based on the properties of new figures—like the horocycle—and certain limits. In this paper I give a unified derivation of some of the important relations of spherical and hyperbolic trigonometry; here a single argument serves both geometries. The basis for such a unification is an insight of E. Beltrami (1835–1906) in whose celebrated paper, Saggio di interpretazione della geometria non-euclidea [2], [18], the first model of non-Euclidean geometry appeared, paving the way for its acceptance.

This framework offers more than the trigonometric formulas: The basic theorems of spherical and non-Euclidean geometry can be arrived at through the same argument, for example, the fact that the medians of a triangle are concurrent, or that the area of a triangle is proportional to the difference between the angle sum and two right angles. Perhaps the most surprising offshoot of this approach is a proof of the irrationality of both π and e from the same source—a slight generalization of the argument of J. Lambert (1728–1777) to include spherical and non-Euclidean trigonometry.

2. BASIC NOTIONS. To better understand Beltrami's work, we review how to describe an abstract surface. The basic example is the sphere in $\mathbb{R}^3$ of radius $R > 0$ centered at the origin,

$$S_R^2 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = R^2\},$$

which we can take as an abstract model of the surface of the Earth. A **map projection** is a mapping X defined on U an open subset of S_R^2, $X\colon U \subset S_R^2 \longrightarrow \mathbb{R}^2$, that is differentiable, one-to-one, and possesses a differentiable inverse. The Mercator projection is the most familiar example. Map projections are well studied and fascinating (see, for example, [12], [13, chapter 8bis]).

More generally, if S is any surface and U is an open subset of S, then a map projection of U is a mapping $X\colon U \to \mathbb{R}^2$ with the same properties. The image of a map projection may be regarded as a cartographic map of a portion of the surface, a page in an atlas of projections that cover the surface.

In order to deduce geometric facts about a surface S, we can investigate the flattened out version of part of the surface, that is, the subset $X(U)$ of $\mathbb{R}^2$. Using the map projection, we project the geometric features of U onto the plane. In this manner it is possible to recover the important measures of figures on a surface, such as lengths of curves, angles between curves, and areas of regions.

The most basic geometric measure along a curve is length. On cartographic map projections of the Earth there are often scales pictured that are meant to allow one to relate distance on the map to distance on the Earth. In the case of the Mercator projection, this scaling is very dependent on latitude, and may be depicted as in Figure 1.

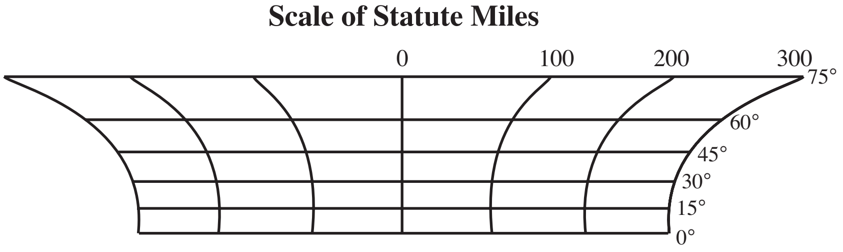

Figure 1.

A more general "scale" is given by the *metric* associated to the image of the projection. The metric assigns an inner product to each point of $X(U)$ with which the length of tangent vectors and the angle between tangent vectors to curves in the plane are determined, as if they were the corresponding curves on the surface and measured there.

A metric is usually presented by an expression

$$ds^2 = E(u,v)du^2 + 2F(u,v)\,du\,dv + G(u,v)dv^2,$$

by which one means that the length of a curve $\alpha\colon [t_0, t_1] \to U$ is given by the integral

$$\int_{t_0}^{t_1} \sqrt{E(u(t),v(t))\left(\frac{du}{dt}\right)^2 + 2F(u(t),v(t))\frac{du}{dt}\frac{dv}{dt} + G(u(t),v(t))\left(\frac{dv}{dt}\right)^2}\,dt,$$

where $(u(t), v(t)) = X \circ \alpha(t)$. When the surface is a subset of $\mathbb{R}^3$, the component functions of the metric are calculable from a map projection by taking certain dot products in $\mathbb{R}^3$:

$$E(u,v) = \frac{\partial X^{-1}}{\partial u} \cdot \frac{\partial X^{-1}}{\partial u}, \quad F(u,v) = \frac{\partial X^{-1}}{\partial u} \cdot \frac{\partial X^{-1}}{\partial v}, \quad G(u,v) = \frac{\partial X^{-1}}{\partial v} \cdot \frac{\partial X^{-1}}{\partial v}.$$

This works for map projections of the sphere.

To describe an *abstract surface* we simply eliminate the domain of the map projection. That is, we are given a collection (the atlas) of coordinate charts (map projections) together with metrics (scales) that determine the geometry locally. Of course, on overlapping pages of the atlas, the metrics must agree, a condition that is made precise by the transformation rules of differential geometry.

For our story, we introduce a particular map projection of the sphere S_R^2. It is defined on the lower hemisphere $U = \{(x,y,z) \in S_R^2 \mid z < 0\}$. We project U onto the tangent plane T to the sphere at its South Pole, $T = \{(x,y,-R) \in \mathbb{R}^3\}$, by passing a line from the center of the sphere through a point in U and extending the line to meet T (Figure 2). This projection is called *central projection* (or *gnomic projection*) and is given in coordinates by

$$X(\lambda, \phi) = (-R\cos(\lambda)\cot(\phi), -R\sin(\lambda)\cot(\phi)),$$

where we have identified T with $\mathbb{R}^2$ by ignoring the z-coordinate, and we have given the sphere its "geographic" coordinates; λ in $(-\pi, \pi)$ denotes the longitude and ϕ in $(-\pi/2, 0)$ denotes the latitude.

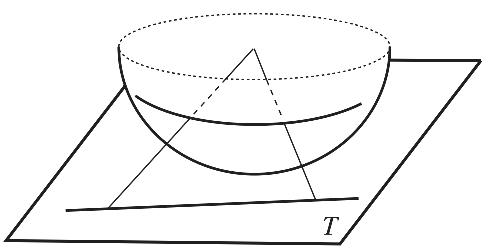

Figure 2.

Central projection enjoys a lovely property: the geodesics on the sphere (the great circles that are the intersection of the sphere with a plane through the origin) are carried to straight lines in the plane. This makes navigation particularly easy using this map projection—the path of least distance on the sphere corresponds to a straight line path on the map. Beltrami posed the problem of determining the surfaces for which such a projection taking geodesics to straight lines exists. He proved that such a surface must have constant curvature [1]. (For a thorough discussion of curvature of surfaces, see [15] or [17].) When the curvature is positive, the spheres S^2_R are the canonical examples.

The metric that arises from central projection is given by

$$ds^2 = R^2 \frac{(R^2+v^2)du^2 - 2uv\,du\,dv + (R^2+u^2)dv^2}{(R^2+u^2+v^2)^2}.$$

If we let $q = 1/R^2$ (the curvature), then ds^2 takes the form

$$ds^2 = \frac{(1+qv^2)du^2 - 2quv\,du\,dv + (1+qu^2)dv^2}{(1+qu^2+qv^2)^2}. \tag{1}$$

In this form an extraordinary opportunity presents itself that was explored by Beltrami in the "Saggio": let q vary over the real numbers. When $q > 0$, the geometry we obtain is the geometry of the sphere of radius $1/\sqrt{q}$. When $q = 0$, we have $ds^2 = du^2 + dv^2$, the Euclidean metric. However, when $q < 0$, Beltrami got something new, a map projection that did not correspond to a concrete surface.

Definition 1. The *q-plane* is the subset $\mathbb{D}_q$ of $\mathbb{R}^2$ given by

$$\mathbb{D}_q = \{(u,v) \mid 1 + qu^2 + qv^2 > 0\},$$

endowed with the metric given by (1).

When $q < 0$, this is the open disk of radius $1/\sqrt{-q}$ centered at $(0,0)$: for $q \geq 0, \mathbb{D}_q = \mathbb{R}^2$. For any choice of q, the Euclidean straight lines in $\mathbb{D}_q$ correspond to geodesics in the geometry determined by ds^2. Our goal is to study trigonometry in $\mathbb{D}_q$, and so we begin with the right triangles.

3. LENGTHS. The basic problem of trigonometry is to determine the relations among the angles and lengths of the sides of a triangle. Spherical trigonometry was first studied by ancient astronomers, making it some of the oldest developed mathematics [16]. Euler [7], [8] gave a complete solution of the general problem of specifying all of the angles and sides of a spherical triangle from any three of the data, a problem known as *solving a spherical triangle*.

To obtain the lengths of the sides of a right triangle, we use the metric ds. In what follows we will assume that the behavior of triangles is the same everywhere in $\mathbb{D}_q$. (This assertion may be proved by showing that there are sufficiently many isometries of $\mathbb{D}_q$ [13, chapter 15].) Thus we can choose a triangle at a particularly convenient place and still deduce universal relations. The convenient place for a right triangle is at the origin $(0,0)$ in $\mathbb{D}_q$ with one leg along the u-axis, as in Figure 3.

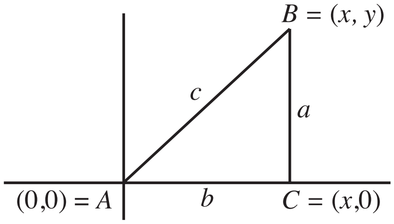

Figure 3.

We first notice that this choice of line segments constitutes a right triangle in $\mathbb{D}_q$. The angle at C is made up of coordinate curves, that is, curves of the form $u \mapsto (u, 0)$ and $v \mapsto (x, v)$. The angle between such curves may be read off the metric: the cosine of the angle between coordinate curves is given by $F/\sqrt{EG}$ [13, p. 112] and $F = -quv/(1 - u^2 - v^2)$. Since $v = 0$ at C, these curves meet at right angles.

We next determine the lengths of the sides. Parametrize AC as the curve $\beta(t) = (t, 0)$ for $0 \leq t \leq x$. Then $d\beta/dt = (1, 0)$, $ds = (1 + qt^2)^{-1}dt$, and we have

$$b = \int_0^x \frac{1}{1 + qt^2}\, dt.$$

For the side BC, let $\alpha(t) = (x, t)$ for $0 \leq t \leq y$. Then $d\alpha/dt = (0, 1)$ and $ds = \sqrt{1 + qx^2}\, dt/(1 + qx^2 + qt^2)$. We then have:

$$a = \int_0^y \frac{\sqrt{1 + qx^2}}{1 + qx^2 + qt^2}\, dt = \int_0^y \frac{d\left(\dfrac{t}{\sqrt{1 + qx^2}}\right)}{1 + q\left(\dfrac{t}{\sqrt{1 + qx^2}}\right)^2} = \int_0^{y/\sqrt{1+qx^2}} \frac{1}{1 + qt^2}\, dt.$$

Finally, for the side AB, let $\gamma(t) = (tx, ty)$ with $0 \leq t \leq 1$. Then $d\gamma/dt = (x, y)$ and

$$\begin{aligned} c &= \int_0^1 \sqrt{\frac{(1 + qt^2y^2)x^2 - 2qt^2x^2y^2 + (1 + qt^2x^2)y^2}{(1 + qt^2x^2 + qt^2y^2)^2}}\, dt \\ &= \int_0^1 \frac{\sqrt{x^2 + y^2}\, dt}{1 + q(x^2 + y^2)t^2} = \int_0^1 \frac{d(\sqrt{x^2 + y^2}\, t)}{1 + q(\sqrt{x^2 + y^2}\, t)^2} = \int_0^{\sqrt{x^2+y^2}} \frac{1}{1 + qt^2}\, dt. \end{aligned}$$

In each case the integral $\displaystyle\int_0^r dt/1 + qt^2$ appears. Following the example of the theory of elliptic functions [20], we introduce a new function.

Definition 2. Let s be a real number. Define the function $\tau_q(s)$ implicitly by the equation

$$s = \int_0^{\tau_q(s)} \frac{1}{1 + qt^2}\, dt.$$

For the right triangle $\triangle ABC$, the definition gives

$$\tau_q(b) = x, \quad \tau_q(a) = \frac{y}{\sqrt{1 + qx^2}}, \quad \tau_q(c) = \sqrt{x^2 + y^2}. \tag{2}$$

The function $\tau_q(s)$ plays the role of the tangent function in $\mathbb{D}_q$. We introduce the analogues of the other trigonometric functions by exploring the properties of $\tau_q(s)$. The first property follows from the fundamental theorem of the calculus:

$$\frac{d\tau_q(s)}{ds} = 1 + q\tau_q^2(s). \tag{3}$$

By analogy with $d/ds(\tan(s)) = \sec^2(s)$, we introduce the following functions to play the role of the cosine and sine functions.

Definition 3. For any real number s, define the functions $\xi_q(s)$ and $\sigma_q(s)$ by

$$\xi_q(s) = \frac{1}{\sqrt{1 + q\tau_q^2(s)}}; \qquad \sigma_q(s) = \tau_q(s)\xi_q(s).$$

We observe from the definitions that

$$1 = \xi_q^2(s)(1 + q\tau_q^2(s)) = \xi_q^2(s) + q\sigma_q^2(s). \tag{4}$$

The Pythagorean Theorem. *Suppose $\triangle ABC$ is a right triangle in $\mathbb{D}_q$ with its right angle at C. If the lengths of the sides opposite the vertices labeled with the corresponding upper case letters are denoted by a, b, and c, then*

$$\xi_q(c) = \xi_q(a)\xi_q(b).$$

Proof. We compute from the definitions:

$$\xi_q(c) = \frac{1}{\sqrt{1 + q\tau_q^2(c)}} = \frac{1}{\sqrt{1 + q(x^2 + y^2)}}.$$

Since

$$1 + qx^2 + qy^2 = (1 + qx^2)\left(1 + q\left(\frac{y}{\sqrt{1 + qx^2}}\right)^2\right) = \frac{1}{\xi_q^2(b)\xi_q^2(a)},$$

it follows that $\xi_q(c) = \xi_q(a)\xi_q(b)$. ■

To relate this expression to the Euclidean Pythagorean theorem, we consider what happens when $q \to 0$. Using infinite series, we can write

$$b = \int_0^x \frac{dt}{1 + qt^2} = \int_0^x \left[1 - qt^2 + q^2t^4 - \cdots\right]\, dt = x - q\frac{x^3}{3} + q^2\frac{x^5}{5} - \cdots.$$

Therefore, $b^2 = x^2 - q(2x^4/3 + \cdots)$. From $a = \displaystyle\int_0^{y/\sqrt{1+qx^2}} dt/1 + qt^2$, we get a similar expansion,

$$a^2 = \frac{y^2}{1 + qx^2} - q\left(\frac{2y^4}{3(1 + qx^2)^2} + \cdots\right)$$

and $c^2 = x^2 + y^2 + q((x^2 + y^2)^{3/2}/3 + \cdots)$. As $q \to 0$, $a^2 + b^2 \to x^2 + y^2$ and $c^2 \to x^2 + y^2$, and so in the limit $c^2 = a^2 + b^2$.

4. FURTHER RELATIONS. In the q-plane for $q \neq 0$ we can solve the problem set by the ancient astronomers for spherical triangles: From three of the pieces of data $\angle A$, $\angle B$, $\angle C$, a, b, and c, determine the other three. Euler [7], [8] gave formula after formula to solve the various cases of the problem. We recover some of the analogous formulas for triangles in the q-plane.

The ordinary trigonometric functions make an appearance by considering the metric at the origin. Here $ds^2 = du^2 + dv^2$. It follows that angles have the same measure as they would if they were in the usual Euclidean plane. For the right triangle $\triangle ABC$ of Figure 3 we have $\cos(\angle A) = x/\sqrt{x^2 + y^2} = \tau_q(b)/\tau_q(c)$. Thus, for a right triangle with right angle at C,

$$\tau_q(b) = \cos(\angle A)\tau_q(c). \tag{5}$$

Similarly, we have $\sin(\angle A) = y/\sqrt{x^2 + y^2}$ or

$$\sin(\angle A)\tau_q(c) = y = \frac{y}{\sqrt{1 + qx^2}}\sqrt{1 + qx^2} = \tau_q(a)\frac{1}{\xi_q(b)},$$

that is, $\sin(\angle A)\xi_q(b)\tau_q(c) = \tau_q(a)$. Using the Pythagorean theorem, we can massage this expression into a nicer one:

$$\sin(\angle A) = \frac{\tau_q(a)}{\tau_q(c)\xi_q(b)} = \frac{\tau_q(a)\xi_q(a)}{\tau_q(c)\xi_q(b)\xi_q(a)} = \frac{\tau_q(a)\xi_q(a)}{\tau_q(c)\xi_q(c)} = \frac{\sigma_q(a)}{\sigma_q(c)}. \tag{6}$$

Finally,

$$\tan(\angle A) = \frac{y}{x} = \frac{1}{x}\frac{y}{\sqrt{1+qx^2}}\sqrt{1+qx^2} = \frac{\tau_q(a)}{\tau_q(b)\xi_q(b)} = \frac{\tau_q(a)}{\sigma_q(b)},$$

that is,

$$\tan(\angle A)\sigma_q(b) = \tau_q(a). \tag{7}$$

We next rotate the triangle in $\mathbb{D}_q$ to place the vertex B at the origin. To see how the coordinates change according to the metric on $\mathbb{D}_q$, we rewrite the coordinates of A, B, and C in terms of the trigonometric functions (Figure 4).

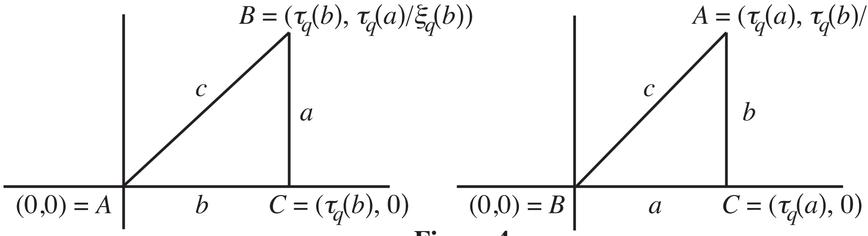

Figure 4.

Since the relations (5)–(7) depend only on the lengths of the sides of the triangle, we deduce

$$\cos(\angle B) = \frac{\tau_q(a)}{\tau_q(c)}, \quad \sin(\angle B) = \frac{\sigma_q(b)}{\sigma_q(c)}, \quad \tan(\angle B) = \frac{\tau_q(b)}{\sigma_q(a)}. \tag{8}$$

Putting together (6) and (8), we arrive at the Law of Sines for right triangles.

Theorem 4. *For a right triangle $\triangle ABC$ with right angle at C in the q-Plane,*

$$\frac{\sin(\angle A)}{\sigma_q(a)} = \frac{\sin(\angle B)}{\sigma_q(b)} = \frac{1}{\sigma_q(c)}.$$

We leave it to the reader to prove the general case of an arbitrary triangle (by choosing a convenient altitude).

Law of Sines. *For an arbitrary triangle $\triangle ABC$ in the q-Plane,*

$$\frac{\sin(\angle A)}{\sigma_q(a)} = \frac{\sin(\angle B)}{\sigma_q(b)} = \frac{\sin(\angle C)}{\sigma_q(c)}.$$

Combining (6) and (7), we can write

$$\begin{aligned}\tan(\angle B) &= \frac{\tau_q(b)}{\sigma_q(a)} = \frac{\tau_q(b)}{\tau_q(a)\xi_q(a)} = \frac{\tau_q(b)}{\tan(\angle A)\sigma_q(b)\xi_q(a)} \\ &= \frac{\cot(\angle A)\tau_q(b)}{\tau_q(b)\xi_q(b)\xi_q(a)} = \frac{\cot(\angle A)}{\xi_q(c)}.\end{aligned}$$

From this relation we infer that

$$\xi_q(c) = \frac{\cot(\angle A)}{\tan(\angle B)} = \frac{\tan(\pi/2 - \angle A)}{\tan(\angle B)}.$$

Since the tangent function is monotonically increasing, whenever $\pi/2 - \angle A > \angle B$, the ratio of their tangents exceeds 1; similarly, if $\pi/2 - \angle A < \angle B$, the ratio of the tangents is less than 1 (I learned this argument from [14]). Thus, the value of $\xi_q(c)$ tells us the sign of $\pi/2 - \angle A - \angle B$. From $\xi_q(c) = 1/\sqrt{1 + qx^2 + qy^2}$ we conclude that $\xi_q(c) < 1$ if $q > 0$ and $\xi_q(c) > 1$ if $q < 0$. This implies that

$$\frac{\pi}{2} + \angle A + \angle B \begin{cases} < \pi, & \text{if } q > 0, \\ > \pi, & \text{if } q < 0. \end{cases}$$

In other words, the angle sum of the interior angles of a right triangle exceeds two right angles if $q > 0$ and falls short of two right angles if $q < 0$.

5. ADDITION FORMULAS.

Among the most important relations enjoyed by trigonometric functions are the addition formulas from which we deduce the double angle and half-angle formulas. Such relations also hold in the q-plane.

Theorem 5. *The function τ_q satisfies the identity*

$$\tau_q(u + v) = \frac{\tau_q(u) + \tau_q(v)}{1 - q\tau_q(u)\tau_q(v)}.$$

Proof. Let $u + v = c$, a constant, and $x = \tau_q(u), y = \tau_q(v) = \tau_q(c - u)$. Then

$$\frac{dx}{du} = 1 + qx^2, \qquad \frac{dy}{du} = -(1 + qy^2).$$

It follows that

$$\frac{dx}{du} + \frac{dy}{du} = q(x^2 - y^2),$$

and so

$$\frac{d}{du}(\ln(x + y)) = \frac{\dfrac{dx}{du} + \dfrac{dy}{du}}{x + y} = q(x - y).$$

Taking another page out of the development of elliptic functions [20], consider

$$y\frac{dx}{du} + x\frac{dy}{du} = y + qx^2y - x - qxy^2 = (y - x) + qxy(x - y) = (y - x)(1 - qxy).$$

From this equation we obtain

$$\frac{d}{du}(\ln(1 - qxy)) = \frac{-q\left(y\dfrac{dx}{du} + x\dfrac{dy}{du}\right)}{1 - qxy} = q(x - y).$$

Thus

$$\frac{d}{du}\left(\ln\left(\frac{x + y}{1 - qxy}\right)\right) = 0.$$

When $x = 0, u = 0$ and so $(x + y)/(1 - qxy) = \tau_q(c)$. The theorem follows. ∎

We leave it to the reader to deduce the corresponding formulas for $\xi_q(s)$ and $\sigma_q(s)$:

$$\xi_q(u + v) = \xi_q(u)\xi_q(v) - q\sigma_q(u)\sigma_q(v), \qquad \sigma_q(u + v) = \sigma_q(u)\xi_q(v) + \sigma_q(v)\xi_q(u).$$

By constructing an altitude inside a triangle that splits it into two right triangles, and applying the addition formulas, the reader can also prove the Law of Cosines in the q-plane.

Law of Cosines. *Let $\triangle ABC$ be an arbitrary triangle in the q-plane with sides of lengths a, b, and c, opposite the vertices A, B, and C, respectively. Then*

$$\xi_q(a) = \xi_q(b)\xi_q(c) + q\sigma_q(b)\sigma_q(c)\cos(\angle A).$$

The addition formula may be applied to obtain both a doubling formula and a halving formula: From $\tau_q(2u) = 2\tau_q(u)/(1 - q\tau_q^2(u))$, we deduce that

$$\tau_q\left(\frac{u}{2}\right) = \frac{-1 + \sqrt{1 + q\tau_q^2(u)}}{q\tau_q(u)}. \tag{9}$$

The halving formula is important in proving the following theorem.

Theorem 6. *The medians of a triangle in the q-plane are concurrent.*

Proof. We prove the result for a right triangle and leave the arbitrary case to the reader. The strategy is to determine the coordinates of the midpoints of the sides of the triangle in the q-plane. The medians in the q-plane are line segments joining each vertex to the midpoint on the opposite side as seen in the triangle lying in the coordinate plane (Figure 5). We can then check the criterion for concurrence given by Ceva's theorem [5] to see that the line segments all pass through the same point from the Euclidean viewpoint.

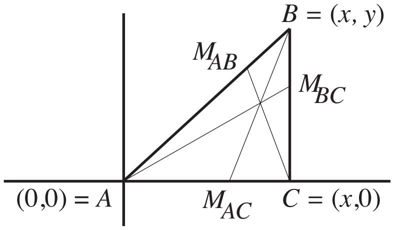

Figure 5.

Following the length computations in section 3, the midpoint of segment AC is $M_{AC} = (\tau_q(b/2), 0)$. The midpoint of segment BC is $M_{BC} = (\tau_q(b), \tau_q(a/2)\xi_q(b))$; this follows from the fact that the right triangle $\triangle AM_{BC}C$ has height $a/2$. The coordinates of the third midpoint, M_{AB}, are a little bit harder to determine. As before, we let $\gamma(t) = (xt, yt)$ and integrate from 0 to u where

$$\frac{c}{2} = \int_0^u \sqrt{\frac{(1 + qt^2y^2)x^2 - 2qt^2x^2y^2 + (1 + qt^2x^2)y^2}{(1 + qt^2x^2 + qt^2y^2)^2}}\, dt = \int_0^{u\sqrt{x+y^2}} \frac{dt}{1 + qt^2},$$

by the same change of variable used earlier to derive the expression for c. Thus $\tau_q(c/2) = u\sqrt{x^2 + y^2} = u\tau_q(c)$, so we have the coordinates of the midpoint of AB,

$$M_{AB} = (xu, yu) = \left(x\frac{\tau_q(c/2)}{\tau_q(c)}, y\frac{\tau_q(c/2)}{\tau_q(c)}\right).$$

To apply Ceva's theorem, we compute the quotient of the *Euclidean* lengths:

$$\frac{AM_{AC}}{M_{AC}C}\,\frac{CM_{BC}}{M_{BC}B}\,\frac{BM_{AB}}{M_{AB}A}.$$

It is easy to evaluate the quotient

$$\frac{AM_{AC}}{M_{AC}C} = \frac{\tau_q(b/2)}{\tau_q(b) - \tau_q(b/2)}.$$

The coordinates of B are given by $(x, y) = (\tau_q(b), \tau_q(a/2)/\xi_q(b))$. Thus

$$\frac{CM_{BC}}{M_{BC}B} = \frac{\tau_q(a/2)/\xi_q(b)}{\tau_q(a)/\xi_q(b) - \tau_q(a/2)/\xi_q(b)} = \frac{\tau_q(a/2)}{\tau_q(a) - \tau_q(a/2)}.$$

Finally, we determine the ratio

$$\frac{BM_{AB}}{M_{AB}A} = \frac{d((x,y),(xu,yu))}{d((0,0),(xu,yu))} = \frac{(1-u)\sqrt{x^2+y^2}}{u\sqrt{x^2+y^2}} = \frac{1-u}{u} = \frac{\tau_q(c) - \tau_q(c/2)}{\tau_q(c/2)}.$$

Because of its ubiquity, we use the halving formula (9) to rewrite quotients of the relevant type:

$$\begin{aligned}\frac{\tau_q(s/2)}{\tau_q(s) - \tau_q(s/2)} &= \frac{\dfrac{-1+\sqrt{1+q\tau_q^2(s)}}{q\tau_q(s)}}{\tau_q(s) - \left(\dfrac{-1+\sqrt{1+q\tau_q^2(s)}}{q\tau_q(s)}\right)} = \frac{-1+\sqrt{1+q\tau_q^2(s)}}{1+q\tau_q^2(s) - \sqrt{1+q\tau_q^2(s)}} \\ &= \frac{1}{\sqrt{1+q\tau_q^2(s)}} = \xi_q(s).\end{aligned}$$

We can then compute the desired product of ratios:

$$\begin{aligned}\frac{AM_{AC}}{M_{AC}C}\frac{CM_{BC}}{M_{BC}B}\frac{BM_{AB}}{M_{AB}A} &= \frac{\tau_q(b/2)}{\tau_q(b) - \tau_q(b/2)}\frac{\tau_q(a/2)}{\tau_q(a) - \tau_q(a/2)}\frac{\tau_q(c) - \tau_q(c/2)}{\tau_q(c/2)} \\ &= \xi_q(a)\xi_q(b)\frac{1}{\xi_q(c)} = 1,\end{aligned}$$

by the Pythagorean theorem. Thus the line segments AM_{BC}, BM_{AC}, and CM_{AB} are concurrent by Ceva's theorem. ■

6. AREA. In order to compute the area of a figure on a surface, we can project it by a map projection and then compute the integral of the function $\sqrt{EG - F^2}$ over the image of the figure [13, p. 113]. In the case of the q-plane, we can compute the area of the right triangle $\triangle ABC$ in this manner. To facilitate the computation of the integral, we change to polar coordinates. The familiar $u = r\cos\theta$ and $v = r\sin\theta$ lead to the following form of the metric:

$$ds^2 = \frac{(1+qv^2)du^2 - 2quv\,du\,dv + (1+qu^2)dv^2}{(1+qu^2+qv^2)^2} = \frac{dr^2}{(1+qr^2)^2} + \frac{r^2d\theta^2}{1+qr^2}.$$

In this case $\sqrt{EG - F^2} = \sqrt{r^2/(1+qr^2)^3}$. With the aid of Figure 6, we can set up the integral that computes the area of $\triangle ABC$:

$$\text{area}(\triangle ABC) = \int_0^{\angle A}\int_0^{x\sec\theta}\sqrt{\frac{r^2}{(1+qr^2)^3}}\,dr\,d\theta.$$

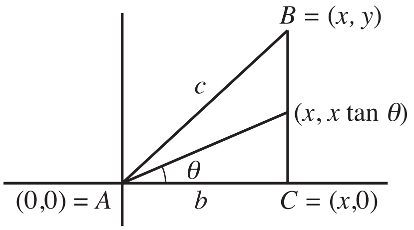

Figure 6.

We compute

$$\begin{aligned}\text{area}(\triangle ABC) &= \int_0^{\angle A} \int_0^{x \sec \theta} \frac{r}{(1+qr^2)^{3/2}} dr\, d\theta = \int_0^{\angle A} -\frac{1}{q}\frac{1}{\sqrt{1+qr^2}}\Bigg|_0^{x \sec \theta} d\theta \\ &= \int_0^{\angle A} \left(\frac{1}{q} - \frac{1}{q\sqrt{1+qx^2\sec^2\theta}}\right) d\theta.\end{aligned}$$

To complete the calculation, we first reorganize the integrand as follows:

$$\begin{aligned}\left(\frac{1}{q} - \frac{1}{q\sqrt{1+qx^2\sec^2\theta}}\right) d\theta &= \left(\frac{1}{q} - \frac{1}{q}\frac{\cos\theta}{\sqrt{1+qx^2-\sin^2\theta}}\right) d\theta \\ &= \frac{d\theta}{q} - \frac{1}{q}\frac{d(\sin\theta)}{\sqrt{1+qx^2-\sin^2\theta}} = \frac{d\theta}{q} - \frac{1}{q}\frac{d(\sin(\theta)/\sqrt{1+qx^2})}{\sqrt{1-(\sin(\theta)/\sqrt{1+qx^2})^2}} \\ &= \frac{d\theta}{q} - \frac{1}{q}\frac{d(\xi_q(b)\sin(\theta))}{\sqrt{1-(\xi_q(b)\sin(\theta))^2}}.\end{aligned}$$

We thus find that

$$\text{area}(\triangle ABC) = \frac{\angle A}{q} - \frac{1}{q}\arcsin(\sin(\angle A)\xi_q(b)).$$

By the identities (6) and (8), we can replace $\sin(\angle A)\xi_q(b)$ by $\tau_q(a)/\tau_q(c) = \cos(\angle B)$. Therefore,

$$\begin{aligned}\text{area}(\triangle ABC) &= \frac{1}{q}(\angle A - \arcsin(\cos(\angle B))) \\ &= \frac{1}{q}(\angle A + \angle B - \pi/2) = \frac{1}{q}\left(\frac{\pi}{2} + \angle A + \angle B - \pi\right).\end{aligned}$$

The area of a right triangle in the q-plane is given by $1/q$ times the angle excess, which is the sum of the interior angles less two right angles. This was proved by Euler [7] for spherical geometry using infinitesimal arguments, while Gauss [6] proved it for non-Euclidean geometry using an ingenious argument involving ideal triangles. In the q-plane, it is a matter of evaluating an integral. The reader is invited to supply the analogous area formula for general triangles.

7. PARTICULAR VALUES. The reader will already recognize $\tau_1(s) = \tan s$ from the integral $\int_0^r dt/1+t^2 = \arctan(r)$. In fact, the differential equations

$$\xi_q'(s) = -q\sigma_q(s), \xi_q(0) = 1; \quad \sigma_q'(s) = \xi_q(s), \sigma_q(0) = 0;$$

determine $\xi_q(s)$ and $\sigma_q(s)$ completely. In the case $q = 1$ we obtain the classical trigonometric functions, and for $q = -1$ the hyperbolic trigonometric functions $\xi_{-1}(s) = \cosh s$ and $\sigma_{-1}(s) = \sinh s$.

We can summarize what we have proved in a table. Let $\triangle ABC)$ be a right triangle with the right angle at C in the q-plane, and $q = 1$ or $q = -1$:

Spherical trigonometry ($q = 1$)	**Hyperbolic trigonometry** ($q = -1$)
$\cos(c) = \cos(a)\cos(b)$	$\cosh(c) = \cosh(a)\cosh(b)$
$\tan(b) = \cos(\angle A)\tan(c)$	$\tanh(b) = \cos(\angle A)\tanh(c)$
$\sin(\angle A)\cos(b)\tan(c) = \tan(a)$	$\sin(\angle A)\cosh(b)\tanh(c) = \tanh(a)$
$\tan(\angle A)\sin(b) = \tan(a)$	$\tan(\angle A)\sinh(b) = \tanh(a)$
$\dfrac{\sin(\angle A)}{\sin(a)} = \dfrac{\sin(\angle B)}{\sin(b)} = \dfrac{1}{\sin(c)}$	$\dfrac{\sin(\angle A)}{\sinh(a)} = \dfrac{\sin(\angle B)}{\sinh(b)} = \dfrac{1}{\sinh(c)}$
area$(\triangle ABC) = \angle A + \angle B - \pi/2$	area$(\triangle ABC) = \pi/2 - \angle A - \angle B$
$\cos(a) = \cos(b)\cos(c) + \sin(b)\sin(c)\cos(\angle A)$	$\cosh(a) = \cosh(b)\cosh(c) - \sinh(b)\sinh(c)\cos(\angle A)$

We have derived each pair of formulas for spherical and hyperbolic trigonometry with a single argument. To find all of the relations that constitute spherical and hyperbolic trigonometry, I encourage the reader to look at Euler [7] and Lobachevsky [3].

We next fix $q < 0$ to prove a key theorem in non-Euclidean geometry. To state the theorem we need a feature of non-Euclidean geometry that distinguishes it from spherical and Euclidean geometry.

Definition 7. Let AC denote a line segment of length s and let CC' denote a line perpendicular to AC through the point C. The *angle of parallelism* is the angle $\Pi(s) = \angle CAX$ formed by the segment AC with a line AX that is parallel to the line CC'. In this case, parallel means that any other line AX' through A making an angle $\angle CAX'$ smaller than $\angle CAX$ will meet CC'.

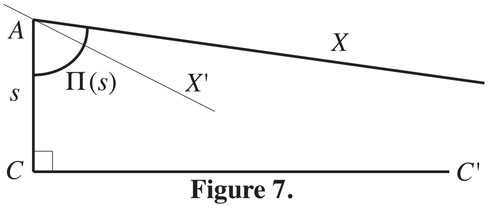

Figure 7.

In Figure 7, the line AX' would meet the line CC'.

In the case $q < 0$, the q-plane is an open disk of radius $1/\sqrt{-q}$. The points on the boundary circle are not in the q-plane, but we may use them as an aid in arguments in hyperbolic geometry. In particular, the angle of parallelism $\Pi(s)$ can be constructed from its definition in the manner of Figure 8:

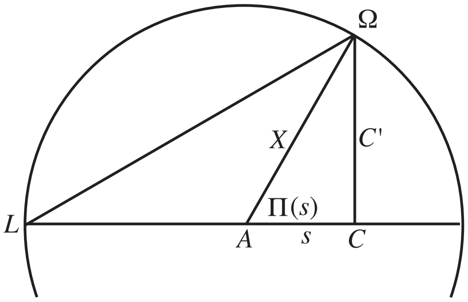

Figure 8.

Here $\angle CA\Omega$ is the angle of parallelism for the length AC. With A at the origin, $\angle CA\Omega$ has the same measure in $\mathbb{D}_q$ as in the Euclidean plane. The Euclidean coordinates of the point Ω are given by $\Omega = (\tau_q(s), \sqrt{(-1/q) - \tau_q^2(s)}\,) = (\tau_q(s), 1/\sqrt{-q}\xi_q(s))$. Let L denote the point $(-1/\sqrt{-q}, 0)$ where the diameter AC meets the boundary circle. According to Euclid (III.20), the Euclidean angle $\angle CL\Omega$ is half of $\angle CA\Omega$. We can now prove the following result [3]:

The Bolyai-Lobachevsky Theorem. *For any $s > 0$,*

$$\tan\left(\frac{\Pi(s)}{2}\right) = e^{-\sqrt{-q}\,s}.$$

Proof. The tangent of the angle $\angle CL\Omega$ may be computed from the Euclidean coordinates:

$$\tan(\angle CL\Omega) = \frac{1/\sqrt{-q}\,\xi_q(s)}{(\tau_q(s) + 1/\sqrt{-q})} = \frac{1}{\xi_q(s) + \sqrt{-q}\sigma_q(s)} = \xi_q(s) - \sqrt{-q}\sigma_q(s).$$

Consider the derivative of the function $\tan(\Pi(s)/2) = \tan(\angle CL\Omega)$:

$$\begin{aligned} \frac{d}{ds}\tan\left(\frac{\Pi(s)}{2}\right) &= \frac{d}{ds}(\xi_q(s) - \sqrt{-q}\sigma_q(s)) \\ &= -q\sigma_q(s) - \sqrt{-q}\xi_q(s) = -\sqrt{-q}(\xi_q(s) - \sqrt{-q}\sigma_q(s)) \\ &= -\sqrt{-q}\tan\left(\frac{\Pi(s)}{2}\right). \end{aligned}$$

Since $\Pi(0) = \pi/2$, $\tan(\Pi(0)/2) = 1$. Accordingly, $\tan(\Pi(s)/2)$ satisfies the differential equation $dy/ds = -\sqrt{-q}\,y$ with initial conditions $y(0) = 1$, whose unique solution is the function $e^{-\sqrt{-q}\,s}$. ■

8. IRRATIONALITIES. The trigonometric functions $\tau_q(s)$, $\xi_q(s)$, and $\sigma_q(s)$ associated to the q-plane, enjoy power series expansions determined by their derivatives and initial conditions. In particular, from $\xi_q''(s) = -q\xi_q(s)$ and $\sigma_q'(s) = \xi_q(s)$, we learn that

$$\xi_q(s) = 1 - \frac{qs^2}{2!} + \frac{qs^4}{4!} - \frac{qs^6}{6!} + \cdots, \quad \sigma_q(s) = s - \frac{qs^3}{3!} + \frac{qs^5}{5!} - \frac{qs^7}{7!} + \cdots, \tag{10}$$

for all real s.

Following Lambert and Gauss (see [11, chapter 18]), we introduce the following series.

Definition 8. The formal power series $F_c(s)$ is defined for real $c > 0$ by

$$F_c(s) = 1 - \frac{s^2}{c} + \frac{s^4}{2!c(c+1)} - \frac{s^6}{3!c(c+1)(c+2)} + \cdots.$$

Because only even powers appear in the series for $F_c(s)$, we can rewrite (10), for q any nonzero real number, $\xi_q(s) = F_{1/2}(s\sqrt{q}/2)$ and $\sigma_q(s) = sF_{3/2}(s\sqrt{q}/2)$.

Lemma 9. *The family of series $F_c(s)$ satisfies the identity*

$$F_{c+1}(s) - F_c(s) = \frac{s^2}{c(c+1)}F_{c+2}(s).$$

Proof. It suffices to observe that

$$\frac{1}{c(c+1)\cdots(c+n)} - \frac{1}{(c+1)(c+2)\cdots(c+n+1)} = \frac{n+1}{c(c+1)\cdots(c+n+1)}.$$

An immediate consequence of Lemma 9 is the identity

$$\frac{F_c(s)}{F_{c+1}(s)} = 1 - \frac{s^2}{c(c+1)}\frac{F_{c+2}(s)}{F_{c+1}(s)}.$$

From this identity, we can derive a continued fraction expansion for $F_c(s)/F_{c+1}(s)$:

$$\begin{aligned}\frac{F_c(s)}{F_{c+1}(s)} = 1 - \frac{s^2}{c(c+1)}\frac{F_{c+2}(s)}{F_{c+1}(s)} &= 1 - \cfrac{s^2/c(c+1)}{\cfrac{F_{c+1}(s)}{F_{c+2}(s)}} \\ = 1 - \cfrac{s^2/c(c+1)}{1 - \cfrac{s^2}{(c+1)(c+2)}\cfrac{F_{c+3}(s)}{F_{c+2}(s)}} &= 1 - \cfrac{s^2/c(c+1)}{1 - \cfrac{s^2/(c+1)(c+2)}{1 - \cfrac{s^2/(c+2)(c+3)}{1 - \ddots}}}.\end{aligned}$$

We apply this to the trigonometric functions $\xi_q(s)$ and $\sigma_q(s)$; by writing $\tau_q(s)/s = \sigma_q(s)/s\xi_q(s)$ we are led to the following continued fraction expansion:

$$\frac{\tau_q(s)}{s} = \frac{F_{3/2}(\sqrt{q}\,s/2)}{F_{1/2}(\sqrt{q}\,s/2)} = \cfrac{1}{1 - \cfrac{qs^2/4\cdot(1/2)\cdot(3/2)}{1 - \cfrac{qs^2/4\cdot(3/2)\cdot(5/2)}{1 - \cfrac{qs^2/4\cdot(5/2)\cdot(7/2)}{1 - \ddots}}}} = \cfrac{1}{1 - \cfrac{qs^2}{3 - \cfrac{qs^2}{5 - \cfrac{qs^2}{7 - \ddots}}}}$$

Theorem 10. *If s and q are rational, then $\tau_q(s)$ is irrational.*

Proof. If $\tau_q(s)/s$ were rational, then we could write $\tau_q(s)/s = a/b$ and $qs^2 = m/n$. Then

$$\frac{a}{b} = \cfrac{1}{1 - \cfrac{m/n}{3 - \cfrac{m/n}{5 - \cfrac{m/n}{7 - \ddots}}}}.$$

However, no rational number has such an infinite continued fraction expression (see [19, Art. 792]). ∎

Corollary 11. The numbers π and e are irrational.

Proof. Let $q = 1$ and $s = \pi/4$. Then $\tau_1(\pi/4)/\pi/4 = 4/\pi$. If π were rational, this would contradict Theorem 10.

For $q = -1$ and $s = 1$, we have $\tau_{-1}(1) = (e - e^{-1})/(e + e^{-1})$. If e were rational, then $\tau_{-1}(1)$ would be rational as well, contradicting the theorem. ∎

The proof of the corollary places the irrationality of π and e into a common framework—each is irrational for trigonometric reasons—π in spherical geometry, e in hyperbolic geometry.

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