6. Compactness

... compact sets play the same role in the theory of abstract sets as the notion of limit sets do in the theory of point sets.
Maurice Frechet, 1906

Compactness is one of the most useful topological properties in analysis, although, at first meeting its definition seems somewhat strange. To motivate the notion of a compact space, consider the properties of a *finite* subset $S \subset X$ of a topological space X. Among the consequences of finiteness are the following:

i) Any continuous function $f\colon X \to \mathbb{R}$, when restricted to S, has a maximum and a minimum.
ii) Any collection of open subsets of X whose union contains S has a finite subcollection whose union contains S.
iii) Any sequence of points $\{x_i\}$ satisfying $x_i \in S$ for all i, has a convergent subsequence.

Compactness extends these properties to other subsets of a space X, using the topology to achieve what finiteness guarantees. The development in this chapter runs parallel to that of Chapter 5 on connectedness.

Definition 6.1. *Given a topological space X and a subset $K \subset X$, a collection of subsets $\{C_i \subset X \mid i \in J\}$ is a* **cover** *of K if $K \subset \bigcup_{i \in J} C_i$. A cover is an* **open cover** *if every C_i is open in X. The cover $\{C_i \mid i \in J\}$ of K has a* **finite subcover** *if there are members of the collection $C_{i_1}, \ldots, C_{i_n}$ with $K \subset C_{i_1} \cup \ldots \cup C_{i_n}$. A subset $K \subset X$ is* **compact** *if* any *open cover of K has a finite subcover.*

Examples: Any finite subset of a topological space is compact. The space $(\mathbb{R}, \text{usual})$ is not compact since the open cover $\{(-n, n) \mid n = 1, 2, \ldots\}$ has no finite subcover. Notice that if K is a subset of $\mathbb{R}^n$ and K is compact, it is *bounded*, that is, $K \subset B(\vec{0}, M)$ for some $M > 0$. This follows since $\{B(\vec{0}, N) \mid N = 1, 2, \ldots\}$ is an open cover of $\mathbb{R}^n$ and hence, of K. Since $B(\vec{0}, N_1) \subset B(\vec{0}, N_2)$ for $N_1 \leq N_2$, a finite subcover is contained in a single open ball and so K is bounded. The canonical example of a compact space is the unit interval $[0, 1] \subset \mathbb{R}$.

The Heine-Borel Theorem. *The closed interval $[0, 1]$ is a compact subspace of $(\mathbb{R}$, usual).*

Proof: Suppose $\{U_i \mid i \in J\}$ is an open cover of $[0, 1]$. Define $T = \{x \in [0, 1] \mid [0, x]$ has a finite subcover from $\{U_i\}\}$. Certainly $0 \in T$ since $0 \in \bigcup U_i$ and so in some U_j. We show $1 \in T$. Since every element of T is less than or equal to 1, T has a least upper bound s. Since $\{U_i\}$ is a cover of $[0, 1]$, for some $j \in J$, $s \in U_j$. Since U_j is open, for some $\epsilon > 0$, $(s - \epsilon, s + \epsilon) \subset U_j$. Since s is a least upper bound, $s - \delta \in T$ for some $0 < \delta < \epsilon$ and so $[0, s - \delta]$ has a finite subcover. It follows that $[0, s]$ has a finite subcover by simply adding U_j to the finite subcover of $[0, s - \delta]$. If $s < 1$, then there is an $\eta > 0$ with $s + \eta \in (s - \epsilon, s + \epsilon) \cap [0, 1]$, and so $s + \eta \in T$, which contradicts the fact that s is a least upper bound. Hence $s = 1$. ♢

Is compactness a topological property? We prove a result analogous to Theorem 5.2 for connectedness.

Proposition 6.2. *If $f\colon X \to Y$ is a continuous function and X is compact, then $f(X) \subset Y$ is compact.*

Proof: Suppose $\{U_i \mid i \in J\}$ is an open cover of $f(X)$ in Y. Then $\{f^{-1}(U_i) \mid i \in J\}$ is an open cover of X. Since X is compact, there is a finite subcover, $\{f^{-1}(U_{i_1}), \ldots, f^{-1}(U_{i_n})\}$. Then $X = f^{-1}(U_{i_1}) \cup \cdots \cup f^{-1}(U_{i_n})$ and so $f(X) \subset U_{i_1} \cup \cdots \cup U_{i_n}$. $\diamondsuit$

It follows immediately that compactness is a topological property. The closed interval $[a, b] \subset (\mathbb{R}, \text{usual})$ is compact for $a < b$. Since S^1 is the continuous image of $[0, 1]$, S^1 is compact. Notice that compactness distinguishes the open and closed intervals in $\mathbb{R}$. Since $(0, 1)$ is homeomorphic to $\mathbb{R}$ and $\mathbb{R}$ is not compact, $(0, 1) \not\cong [0, 1]$. Since $(0, 1) \subset [0, 1]$, arbitrary subspaces of compact spaces need not be compact. However, compactness is inherited by closed subsets.

Proposition 6.3. *If X is a compact space and $K \subset X$ is a closed subset, then K is compact.*

Proof: If $\{U_i \mid i \in J\}$ is an open cover of K, we can take the collection $\{X-K\} \cup \{U_i \mid i \in J\}$ as an open cover of X. Since X is compact, the collection has a finite subcover, namely $\{X - K, U_{i_1}, \ldots, U_{i_n}\}$. Leaving out $X - K$, we get $\{U_{i_1}, \ldots, U_{i_n}\}$, a finite subcover of K. $\diamondsuit$

A partial converse holds for Hausdorff spaces.

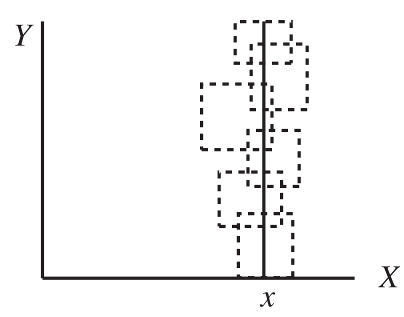
Proposition 6.4. *If X is Hausdorff and $K \subset X$ is compact, then K is closed in X.*

Proof: We show $X - K$ is open. Take $x \in X - K$. By the Hausdorff condition, for each $y \in K$ there are open sets U_y, V_y with $x \in U_y$, $y \in V_y$ and $U_y \cap V_y = \emptyset$. Then $\{V_y \mid y \in K\}$ is an open cover of K. Since K is compact, there is a finite subcover $\{V_{y_1}, V_{y_2}, \ldots, V_{y_n}\}$. Take the associated open sets $U_{y_1}, \ldots, U_{y_n}$ and define $U_x = U_{y_1} \cap \cdots \cap U_{y_n}$. Since $U_{y_i} \cap V_{y_i} = \emptyset$, U_x doesn't meet $V_{y_1} \cup \ldots \cup V_{y_n} \supset K$. So $U_x \subset X - K$. Furthermore, $x \in U_x$ and U_x is open. Construct U_x for every point x in $X - K$, and the union of these open sets U_x is $X - K$ and K is closed. $\diamondsuit$

Corollary 6.5. *If $K \subset \mathbb{R}^n$ is compact, K is closed and bounded.*

Quotient spaces of compact spaces are seen to be compact by Theorem 6.2. The converse of Corollary 6.5 will follow from a consideration of finite products.

Proposition 6.6. *If X and Y are compact spaces, then $X \times Y$ is compact.*



Proof: Suppose $\{U_i \mid i \in J\}$ is an open cover of $X \times Y$. From the definition of the product topology, each $U_i = \bigcup_{j \in A_i} V_{ij} \times W_{ij}$ where V_{ij} is open in X, W_{ij} is open in Y and A_i is an indexing set. Consider the associated open cover $\{V_{ij} \times W_{ij} \mid i \in J, j \in A_i\}$ by basic open sets. If we can manufacture a finite subcover from this collection, we can just take the U_i in which each basic open set sits to get a finite subcover of $X \times Y$.

To each $x \in X$ consider the subspace $\{x\} \times Y \subset X \times Y$. This subspace is homeomorphic to Y and hence is compact. Furthermore $\{V_{ij} \times W_{ij} \mid i \in J,\, j \in A_i\}$ covers $\{x\} \times Y$ and so there is a finite subcover $V_1^x \times W_1^x, \ldots, V_e^x \times W_e^x$ of $\{x\} \times Y$. Let $V^x = V_1^x \cap \cdots \cap V_e^x$. Since $x \in V^x$, it is a nonempty open set. Construct V^x for each $x \in X$ and the collection $\{V^x \mid x \in X\}$ is an open cover of X. Since X is compact, there is a finite subcover, $V^{x_1}, \ldots, V^{x_n}$. Hence each $x \in X$ appears in some V^{x_i}. If $y \in Y$, then $(x, y) \in V_j^{x_i} \times W_j^{x_i}$ for some $W_j^{x_i}$ since $x \in V_1^{x_i} \cap \ldots \cap V_{e_i}^{x_i}$ and $V_1^{x_i} \times W_1^{x_i}, \ldots, V_{e_i}^{x_i} \times W_{e_i}^{x_i}$ covers $\{x\} \times Y$. Hence $\{V_j^{x_i} \times W_j^{x_i} \mid i = 1, \ldots, n,\, j = 1, \ldots, e_i\}$ is a finite subcover of $X \times Y$. The associated choices of U_i's give the finite subcover we seek. ♢

By induction, any finite product of compact spaces is compact. Since $[0,1] \times [0,1]$ is compact, so are the quotients given by the torus, Möbius band and projective plane. We can now prove the converse of Corollary 6.5.

Corollary 6.7. *If $K \subset \mathbb{R}^n$, then K is compact if and only if K is closed and bounded.*

Proof: A bounded subset of $\mathbb{R}^n$ is contained in some product of closed intervals $[a_1, b_1] \times \cdots \times [a_n, b_n]$. The product is compact, and K is a closed subset of $[a_1, b_1] \times \cdots \times [a_n, b_n]$. By Proposition 6.3, K is compact. ♢

We can add the spheres $S^{n-1} \subset \mathbb{R}^n$ to the list of compact spaces—each is bounded by definition and closed because $S^{n-1} = f^{-1}(\{1\})$ where $f \colon \mathbb{R}^n \to \mathbb{R}$ is the continuous function $f(x_1, \ldots, x_n) = x_1^2 + \cdots + x_n^2$. The characterization of compact subsets of $\mathbb{R}$ leads to the following familiar result.

The Extreme Value Theorem. *If $f \colon X \to \mathbb{R}$ is a continuous function and X is compact, then there are points x_m, $x_M \in X$ with $f(x_m) \leq f(x) \leq f(x_M)$ for all $x \in X$.*

Proof: By Proposition 6.2, $f(X)$ is a compact subset of $\mathbb{R}$ and so $f(X)$ is closed and bounded. The boundedness implies that the greatest lower bound of $f(X)$ and the least upper bound of $f(X)$ exist. Since $f(X)$ closed, the values $\operatorname{glb} f(X)$ and $\operatorname{lub} f(X)$ are in $f(X)$ (Can you prove this?) and so $\operatorname{glb} f(X) = f(x_m)$ for some $x_m \in X$; also $\operatorname{lub} f(X) = f(x_M)$ for some $x_M \in X$. It follows that $f(x_m) \leq f(x) \leq f(x_M)$ for all $x \in X$. ♢

The reader might enjoy deriving the whole of the single variable calculus armed with the Intermediate Value Theorem and the Extreme Value Theorem.

Infinite products of compact spaces are covered by the following powerful theorem which turns out to be equivalent to the Axiom of Choice in set theory [Kelley]. We refer the reader to [Munkres] for a proof.

Tychonoff's Theorem. *If $\{X_i \mid i \in J\}$ is a collection of nonempty compact spaces, then, with the product topology, $\prod_{i \in J} X_i$ is compact.*

Infinite products give a different structure in which to consider families of functions with certain properties as subspaces of a product. General products also provide spaces in which there is a lot of room for embedding classes of spaces as subspaces of a product.

Compact spaces enjoy some other interesting properties:

Proposition 6.8. *If $R = \{x_\alpha \mid \alpha \in J\}$ is an infinite subset of a compact space X, then R has a limit point.*

Proof: Suppose R has no limit points. The absence of limit points implies that, for every $x \in X$, there is an open set U^x with $x \in U^x$ for which, if $x \in R$, then $U^x \cap R = \{x\}$ and if $x \notin R$, then $U^x \cap R = \emptyset$. The collection $\{U^x \mid x \in X\}$ is an open cover of X, which is compact, and so it has a finite subcover, $U^{x_1}, \ldots, U^{x_n}$. Since each U^{x_i} contains at most one element in $\{x_\alpha \mid \alpha \in J\}$, the set R is finite. $\diamondsuit$

The property that an infinite subset must have a limit point is sometimes called the *Bolzano-Weierstrass property* [Munkres-red]. The proposition gives a sufficient test for noncompactness: Find a sequence without a limit point. For example, if we give $\prod_{i=1}^{\infty}[0,1]$, the countable product of $[0,1]$ with itself, the box topology, then the set $\{(x_{n,i}) \in \prod_{i=1}^{\infty} \mid n = 1, 2, \ldots\}$ given by $x_{n,i} = 1$ when $n \neq i$ and $x_{n,i} = 1/n$ if $n = i$, has no limit point. (Can you prove it?)

Compactness provides a simple condition for a mapping to be a homeomorphism.

PROPOSITION 6.9. *If $f\colon X \to Y$ is continuous, one-one, and onto, X is compact, and Y is Hausdorff, then f is a homeomorphism.*

Proof: We show that f^{-1} is a continuous by showing that f is closed (that is, $f(K)$ is closed whenever K is closed). If $K \subset X$ is closed, then it is compact. It follows that $f(K)$ is compact in Y and so $f(K)$ is closed because Y is Hausdorff. $\diamondsuit$

Proposition 6.9 can make the comparison of quotient spaces and other spaces easier. For example, suppose X is a compact space with an equivalence relation $\sim$ on it, and $\pi\colon X \to [X]$ is a quotient mapping. Given a mapping $f\colon X \to Y$ for which $x \sim x'$ implies $f(x) = f(x')$, we get an induced mapping $\hat{f}\colon [X] \to Y$ that may be one-one, onto, and continuous. If Y is Hausdorff, we obtain that $[X]$ is homeomorphic to Y.

What about compact metric spaces? The **diameter** of a subset A of a metric space X is defined by $\operatorname{diam} A = \sup\{d(x,y) \mid x, y \in A\}$.

LEBESGUE'S LEMMA. *Let X be a compact metric space and $\{U_i \mid i \in J\}$ an open cover. Then there is a real number $\delta > 0$* (**the Lebesgue number**) *such that any subset of X of diameter less than δ is contained in some U_i.*

Proof: In the exercises to Chapter 3 we defined the continuous function $d(-,A)\colon X \to \mathbb{R}$ by $d(x,A) = \inf\{d(x,a) \mid x \in A\}$. In addition, if A is closed, then $d(x,A) > 0$ for $x \notin A$. Given an open cover $\{U_i \mid i \in J\}$ of the compact space X, there is a finite subcover $\{U_{i_1}, \ldots, U_{i_n}\}$. Define $\varphi_j(x) = d(x, X - U_{i_j})$ for $j = 1, 2, \ldots, n$ and let $\varphi(x) = \max\{\varphi_1(x), \ldots, \varphi_n(x)\}$. Since each $x \in X$ lies in some $U_{i_j}, \varphi(x) \geq \varphi_j(x) > 0$. Furthermore, φ is continuous so $\varphi(X) \subset \mathbb{R}$ is compact, and $0 \notin \varphi(X)$. Let $\delta = \min\{\varphi(x) \mid x \in X\} > 0$. For any $x \in X$, consider $B(x,\delta) \subset X$. We know $\varphi(x) = \varphi_j(x)$ for some j. For that j, $d(x, X - U_{i_j}) \geq \delta$, which implies $B(x,\delta) \subset U_{i_j}$. $\diamondsuit$

The Lebesgue Lemma is also known as the *Pflastersatz* [Alexandroff-Hopf] (imagine plasters covering a space) and it will play a key role in later chapters.

By analogy with connectedness and path-connectedness we introduce the local version of compactness.

DEFINITION 6.10. *A space X is* **locally compact** *if for any $x \in U \subset X$ where U is an open set, there is an open set V satisfying $x \in V \subset \operatorname{cls} V \subset U$ with* cls V *compact.*

Examples: For all n, the space $\mathbb{R}^n$ is locally compact since each cls $B(\vec{x}, \epsilon)$ is compact (being closed and bounded). The countable product of copies of $\mathbb{R}$, $\prod_{i=1}^{\infty} \mathbb{R}$, in the product topology, however, is not locally compact. To see this consider any open set of the form

$$U = (a_1, b_1) \times (a_2, b_2) \times \cdots \times (a_n, b_n) \times \mathbb{R} \times \mathbb{R} \times \ldots$$

whose closure is $[a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_n, b_n] \times \mathbb{R} \times \mathbb{R} \times \cdots$. This set is not compact because there is plenty of room for infinite sets to float off without limit points. Thus local compactness distinguishes finite and infinite products of $\mathbb{R}$, a partial result toward the topological invariance of dimension.

In the presence of local compactness and a little more, we can make a noncompact space into a compact one.

Definition 6.11. *Let X be a locally compact, Hausdorff space. Adjoin a point not in X, denoted by ∞, to form $Y = X \cup \{\infty\}$. Topologize Y by two kinds of open sets:* (1) *$U \subset X \subset Y$ and U is open in X.* (2) *$Y - K$ where K is compact in X. The space Y with this topology is called the* **one-point compactification** *of X.*

The one-point compactification was introduced by Alexandroff [Alexandroff] and is also called the *Alexandroff compactification*. We verify that we have a topology on Y as follows: For finite intersections there are the three cases: We only need to consider the case of two open sets. (1) If V_1 and V_2 are both open subsets of X, then $V_1 \cap V_2$ is also an open subset of X and hence of Y. (2) If both V_1 and V_2 have the form $Y - K_1$ and $Y - K_2$ where K_1 and K_2 are compact in X, then $(Y - K_1) \cap (Y - K_2) = Y - (K_1 \cup K_2)$ and $K_1 \cup K_2$ is compact in X, so $V_1 \cap V_2$ is open in Y. (3) If V_1 is an open subset of X and $V_2 = Y - K_2$ for K_2 compact in X, then $V_1 \cap V_2 = V_1 \cap (Y - K_2) = V_1 \cap (X - K_2)$ since $V_1 \subset X$. Since $X - K_2$ is open in X, the intersection $V_1 \cap V_2$ is open in Y.

For arbitrary unions there are three similar cases. If $\{V_\beta \mid \beta \in I\}$ is a collection of open sets, then $\bigcup V_\beta$ is certainly open when $V_\beta \subset X$ for all β. If $V_\beta = Y - K_\beta$ for all β, then DeMorgan's law gives

$$\bigcup_\beta (Y - K_\beta) = Y - \bigcap_\beta K_\beta$$

and $\bigcap_\beta K_\beta$ is compact. Finally, if the V_β are of different types, the set-theoretic fact $U \cup (Y - K) = Y - (K - U)$ together with the fact that if K is compact, then, since $K - U$ is a closed subset of K, so $K - U$ is compact. Thus the union of the V_β is open in Y.

Theorem 6.12. *If X is locally compact and Hausdorff, X is not compact, and $Y = X \cup \{\infty\}$ is the one-point compactification, then Y is a compact Hausdorff space, X is a subspace of Y, and* cls $X = Y$.

Proof: We first show Y is compact. If $\{V_i \mid i \in J\}$ is an open cover of Y, then $\infty \in V_{j_0}$ for some $j_0 \in J$ and $V_{j_0} = Y - K_{j_0}$ for K_{j_0} compact in X. Since any open set in Y satisfies the property that $V_i \cap X$ is open in X, the collection $\{V_i \cap X \mid i \in J, i \neq j_0\}$ is an open cover of K_{j_0} and so there is a finite subcover $V_1 \cap X, \ldots, V_n \cap X$ of X. Then $\{V_{j_0}, V_1, \ldots, V_n\}$ is a finite subcover of Y.

Next we show Y is Hausdorff. The important case to check is a separation of $x \in X$ and ∞. Since X is locally compact and X is open in X, there is an open set $V \subset X$ with

$x \in V$ and cls V compact. Take V to contain x and $Y - \text{cls } V$ to contain ∞. Since X is not compact, cls $V \neq X$.

Notice that the inclusion $i: X \to Y$ is continuous since $i^{-1}(Y - K) = X - K$ and K is closed in the Hausdorff space X. Furthermore, i is an open map so X is homeomorphic to $Y - \{\infty\}$. To prove that cls $X = Y$, check that ∞ is a limit point of X: if $\infty \in Y - K$, since X is not compact, $K \neq X$ so there is a point of X in $Y - K$ not equal to ∞. ♢

Example: Stereographic projection of the sphere S^2 minus the North Pole onto $\mathbb{R}^2$, shows that the one-point compactification of $\mathbb{R}^2$ is homeomorphic to S^2. Recall that stereographic projection is defined as the mapping from S^2 minus the North Pole to the plane tangent to the South Pole by joining a point on the sphere to the North Pole and then extending this line to meet the tangent plane. This mapping has wonderful properties ([McCleary]) and gives the homeomorphism between $\mathbb{R}^2 \cup \{\infty\}$ and S^2. More generally, $\mathbb{R}^n \cup \{\infty\} \cong S^n$.

Compactness may be used to define a topology on $\text{Hom}(X, Y) = \{f: X \to Y$ such that f is continuous$\}$. There are many possible choices, some dependent on the topologies of X and Y (for example, for metric spaces), some appropriate to the analytic applications for which a topology is needed [Day]. We present one particular choice that is useful for topological applications.

Definition 6.13. *Suppose $K \subset X$ and $U \subset Y$. Let $S(K, U) = \{f: X \to Y$, continuous with $f(K) \subset U\}$. The collection $\mathcal{S} = \{S(K, U) \mid K \subset X$ compact, $U \subset Y$ open$\}$ is a subbasis for topology $\mathcal{T}_{\mathcal{S}}$ on $\text{Hom}(X, Y)$ called the* **compact-open topology**. *We denote the space $(\text{Hom}(X, Y), \mathcal{T}_{\mathcal{S}})$ as $\text{map}(X, Y)$.*

Theorem 6.14. (1) *If X is locally compact and Hausdorff, then the evaluation mapping*

$$e: X \times \text{map}(X, Y) \to Y, \qquad e(x, f) = f(x),$$

is continuous. (2) *If X is locally compact and Hausdorff and Z is another space, then a function $F: X \times Z \to Y$ is continuous if and only if its adjoint map $\hat{F}: Z \to \text{map}(X, Y)$, defined by $\hat{F}(z)(x) = F(x, z)$ is continuous.*

Proof: Given $(x, f) \in X \times \text{map}(X, Y)$ suppose $f(x) \in V$ an open set in Y. Since $x \in f^{-1}(V)$, use the fact that X is locally compact to find U open in X such that $x \in U \subset \text{cls } U \subset f^{-1}(V)$ with cls U compact. Then $(x, f) \in U \times S(\text{cls } U, V)$, an open set of $X \times \text{map}(X, Y)$. If $(x_1, f_1) \in U \times S(\text{cls } U, V)$, then $f_1(x_1) \in V$ so $e(U \times S(\text{cls } U, V)) \subset V$ as needed.

Suppose $\hat{F}$ is continuous. Then F is the composite

$$e \circ (\text{id} \times \hat{F}): X \times Z \to X \times \text{map}(X, Y) \to Y,$$

which is continuous.

Suppose F is continuous and consider $\hat{F}: Z \to \text{map}(X, Y)$. Let $z \in Z$ and $S(K, U)$ a subbasis open set containing $\hat{F}(z)$. We show there is an open set $W \subset Z$, with $z \in W$ and $\hat{F}(W) \subset S(K, U)$. Since $\hat{F}(z) \in S(K, U)$, we have $F(K \times \{z\}) \subset U$. Since F is continuous, it follows that $K \times \{z\} \subset F^{-1}(U)$ and $F^{-1}(U)$ is an open set in $X \times Z$. The subset $K \times \{z\}$ is compact and so the collection of basic open sets contained in $F^{-1}(U) \subset X \times Z$ gives an open cover of $K \times \{z\}$. This cover has a finite subcover, $U_1 \times W_1, U_2 \times W_2, \ldots, U_n \times W_n$. Let

$W = W_1 \cap W_2 \cap \cdots \cap W_n$, a nonempty open set in Z since $z \in W_i$ for each i. Furthermore, $K \times W \subset F^{-1}(U)$. If $z' \in W$ and $x \in K$ then $F(x, z') \in U$, and so $\hat{F}(W) \subset S(K, U)$ as desired. ♢

The description of topology as "rubber-sheet geometry" can be made precise by picturing $\mathrm{map}(X, Y)$. We want to describe a deformation of one mapping into another. If f and g are in $\mathrm{map}(X, Y)$, then a path in $\mathrm{map}(X, Y)$ joining f and g is a mapping $\lambda\colon [0, 1] \to \mathrm{map}(X, Y)$ with $\lambda(0) = f$ and $\lambda(1) = g$. This path encodes the deforming of $f(X)$ to $g(X)$ where at time t the shape is $\lambda(t)(X)$. We can rewrite this path using the adjoint to define an important notion to be developed in later chapters.

Definition 6.15. *A* **homotopy** *between functions* $f, g\colon X \to Y$ *is a continuous function* $H\colon X \times [0, 1] \to Y$ *with* $H(x, 0) = f(x)$, $H(x, 1) = g(x)$. *We say that* f **is homotopic to** g *if there is a homotopy between them.*

Notice that $\hat{H} = \lambda$, a path between f and g in $\mathrm{map}(X, Y)$. A homotopy may be thought of as a continuous, one-parameter family of functions deforming f into g.

We record some other important properties of the compact-open topology. The proofs are left to the reader:

Proposition 6.16. *Suppose that X is a locally compact and Hausdorff space.* (1) *If* $(\mathrm{Hom}(X, Y), \mathcal{T})$ *is another topology on* $\mathrm{Hom}(X, Y)$ *and the evaluation map,*

$$e\colon X \times (\mathrm{Hom}(X, Y), \mathcal{T}) \to Y$$

is continuous, then $\mathcal{T}$ contains the compact-open topology. (2) *If X and Y are locally compact and Hausdorff, then the composition of functions*

$$\circ\colon \mathrm{map}(X, Y) \times \mathrm{map}(Y, Z) \longrightarrow \mathrm{map}(X, Z)$$

is continuous. (3) *If Y is Hausdorff, then the space* $\mathrm{map}(X, Y)$ *is Hausdorff.*

Conditions on continuous mappings from X to Y lead to subsets of $\mathrm{map}(X, Y)$ that may be endowed with the subspace topology. For example, let $\mathrm{map}((X, x_0), (Y, y_0))$ denote the subspace of functions $f\colon X \to Y$ for which $f(x_0) = y_0$. This is the **space of pointed maps**. More generally, if $A \subset X$ and $B \subset Y$, we can define the space of maps of pairs, $\mathrm{map}((X, A), (Y, B))$, requiring that $f(A) \subset B$.

Exercises

1. A second countable space is "almost" compact. Prove that when X is second countable, every open cover of X has a countable subcover (Lindelöf's theorem).

2. Show that a compact Hausdorff space is *normal* (also labelled T_4), that is, given two disjoint closed subsets of X, say A and B, then there are open sets U and V with $A \subset U$, $B \subset V$ and $U \cap V = \emptyset$.

3. A useful property of compact spaces is the **finite intersection property**. Suppose that $\mathcal{F} = \{F_j \mid j \in J\}$ is a collection of closed subsets of X with the following property:

$F_1 \cap \cdots \cap F_k \neq \emptyset$ for every finite subcollection $\{F_1, \ldots, F_k\}$ of $\mathcal{F}$, then $\bigcap_{F \in \mathcal{F}} F \neq \emptyset$. Show that this condition is equivalent to a space being compact. (Hint: Consider the complements of the F_i and the consequence of the intersection being empty.)

4. Suppose X is a compact space and $\{x_1, x_2, x_3, \ldots\}$ is a sequence of points in X. Show that there is a subsequence of $\{x_i\}$ that converges to a point in X.

5. Show the easy direction of Tychonoff's theorem, that is, if $\{X_i \mid i \in J\}$ is a collection of nonempty spaces, and the product $\prod_{i \in J} X_i$ is compact, then each X_i is compact.

6. Although the compact subsets of $\mathbb{R}$ are easily determined (closed and bounded), things are very different in $\mathbb{Q} \subset \mathbb{R}$ with the subspace topology. Determine the compact subsets of $\mathbb{Q}$. We can mimic the one-point compactification of $\mathbb{R}$ using $\mathbb{Q}$: Let $\hat{\mathbb{Q}} = \mathbb{Q} \cup \{\infty\}$ topologized by $T = \{U \subset \mathbb{Q}, U$ open, or $\hat{\mathbb{Q}} - K$, where K is a compact subset of $\mathbb{Q}\}$. Show that $(\hat{\mathbb{Q}}, T)$ is not Hausdorff. Deduce that $\mathbb{Q}$ is not locally compact.

7. Proposition 6.9 states that if $f \colon X \to Y$ is one-one, onto, and continuous, if X is compact, and Y is Hausdorff, then f is a homeomorphism. Show that the condition of Y Hausdorff cannot be relaxed.

8. Prove Proposition 6.16.