

12

Spectral Sequences in Algebra, Geometry and Analysis

“During the last decade the methods of algebraic topology have invaded extensively the domain of pure algebra, and initiated a number of internal revolutions.”

From [Cartan-Eilenberg56]

Spectral sequences arise from filtered differential modules, from double complexes, and from exact couples (Chapter 2). These basic structures may be found in almost any situation where homological methods are used—many examples of spectral sequences have become essential tools in fields outside of topology.

In this chapter, we continue the catalogue begun in Chapter 11. The examples here fall into three broad classes: those of homological origin (§12.1); those based on algebraic or differential geometric structures (§12.2), and those whose origin is chiefly topological but whose interpretation is algebraic (§12.3). We close the chapter with a short discussion of the notion of derived categories (§12.4), a formalism that lurks behind the ‘unreasonable effectiveness’ of spectral sequences.

The reader is expected to be acquainted with the categories of discourse for the examples presented in this chapter—definitions can be found in the cited references. Furthermore, this catalogue is quite far from complete (though some might argue that inclusion of Grothendieck’s composite functor spectral sequence excludes very few examples). The hope remains that the reader will find a useful example in this collection or at least the sense in which spectral sequences can be applied in his or her field of interest. A search of the review literature in mathematics will provide a bounty of details to the curious reader.

12.1 Spectral sequences for rings and modules

Suppose R and S are commutative rings with unit. Denote the category of left (right) modules over R by ${}_R\mathbf{Mod}$ (\mathbf{Mod}_R) and similarly for the ring S . If we are given a homomorphism of rings, $\varphi: R \rightarrow S$, then modules over S obtain the structure of modules over R , where $r \cdot m = \varphi(r) \cdot m$. Under these conditions,

we would like to relate the homological invariants of modules over the given rings in terms of the homomorphism φ . The following theorem describes these relations in terms of spectral sequences due to [Cartan-Eilenberg56].

Theorem 12.1 (the change-of-rings spectral sequences). *Suppose $\varphi: R \rightarrow S$ is a homomorphism of commutative rings with unit, and $M \in \mathbf{Mod}_R$, $N \in {}_S\mathbf{Mod}$, $M' \in \mathbf{Mod}_S$ and $N' \in {}_R\mathbf{Mod}$.*

- (1) *There is a spectral sequence with $E_{p,q}^2 \cong \mathrm{Tor}_p^S(\mathrm{Tor}_q^R(M, S), N)$, and converging to $\mathrm{Tor}_*^R(M, N)$.*
- (2) *There is a spectral sequence with $E_{p,q}^2 \cong \mathrm{Tor}_p^S(M', \mathrm{Tor}_q^R(S, N'))$, and converging to $\mathrm{Tor}_*^R(M', N')$.*
- (3) *There is a spectral sequence with $E_2^{p,q} \cong \mathrm{Ext}_S^q(\mathrm{Tor}_p^R(S, N'), N)$, and converging to $\mathrm{Ext}_R^*(N', N)$.*
- (4) *There is a spectral sequence with $E_2^{p,q} \cong \mathrm{Ext}_S^p(N, \mathrm{Ext}_R^q(S, N'))$, and converging to $\mathrm{Ext}_R^*(N, N')$.*

These spectral sequences can be derived in the manner of Theorem 2.20 (the Künneth spectral sequence) by judicious choices of double complexes. They are also special cases of the Grothendieck spectral sequence for derived functors on abelian categories (Theorem 12.??).

An important special case of the change-of-rings spectral sequence is an extension of algebras over a field, $0 \rightarrow B \rightarrow A \rightarrow A//B \rightarrow 0$.

Theorem 12.2. *Suppose A is an augmented algebra over a field, k . Suppose B is normal subalgebra of A and A is projective over B . If $M \in \mathbf{Mod}_A$ and $N \in {}_{A//B}\mathbf{Mod}$, then there is a spectral sequence with*

$$E_{p,q}^2 \cong \mathrm{Tor}_p^{A//B}(\mathrm{Tor}_q^B(M, k), N)$$

and converging to $\mathrm{Tor}_^A(M, N)$. Also, for $M' \in {}_A\mathbf{Mod}$, there is a spectral sequence with*

$$E_2^{p,q} \cong \mathrm{Ext}_{A//B}^p(N, \mathrm{Ext}_B^q(k, M'))$$

and converging to $\mathrm{Ext}_A^(N, M')$.*

The reader can compare this theorem with Theorem 9.12 for central extensions of Hopf algebras.

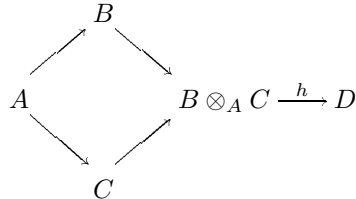
The next three examples represent special cases of extra structures on rings or algebras that lead to spectral sequences. The first bears a strong relation to the Eilenberg-Moore spectral sequence. The second example treats some other homological invariants of a ring, namely the Hochschild homology and the cyclic homology of a ring. The third example is quite general and focuses on the consequences of a filtration on an algebra and a module over it; a special case appears as Theorem 9.56.

In the study of local rings, various homological invariants have played a key role. In particular, the Poincaré series for a local ring, (R, \mathfrak{m}) , is defined by

$$P_R(t) = \sum_{i=0}^{\infty} (\dim_{R/\mathfrak{m}} \operatorname{Tor}_i^R(R/\mathfrak{m}, R/\mathfrak{m}))t^i.$$

Several spectral sequences have been used to study this series ([Avramov-Halperin86]). The following example was derived by [Avramov81] to study the problem of whether a minimal free resolution can be given the structure of an algebra.

Theorem 12.3. *Given a diagram of commutative ring homomorphisms*



and a module M over B , then $\operatorname{Tor}^A(M, C)$ is a $\operatorname{Tor}^A(B, C)$ -module, and D is a $\operatorname{Tor}^A(B, C)$ -module via the homomorphism h . Furthermore, there is a spectral sequence with $E_{p,q}^2 \cong \operatorname{Tor}_{p,q}^{\operatorname{Tor}^A(B,C)}(\operatorname{Tor}^A(M, C), D)$, and converging to $\operatorname{Tor}_*^B(M, D)$.

[Avramov81] used the spectral sequence to build an obstruction theory for the existence of multiplicative structures on resolutions. He also explicated the relationship of this spectral sequence to the Eilenberg-Moore spectral sequence.

An invariant of associative algebras over a fixed ring R was introduced by [Hochschild45] to study the classification of extensions of algebras. One expression for the Hochschild homology of an algebra that is projective as a module over R is given by

$$HH_*(A) \cong \operatorname{Tor}_*^{A^{\text{op}} \otimes A}(A, A),$$

where $A^{\text{op}} \otimes A$ acts on A by $(\alpha \otimes \beta)(a) = \beta a \alpha$. [Hochschild45] gave a functorial complex, resembling the bar construction, to compute $HH_*(A)$. When A is a regular affine k -algebra over a perfect field k , [Hochschild-Konstant-Rosenberg62] proved that $HH_*(A)$ is isomorphic to the algebraic deRham complex $\Omega_{A/k}^* = \Lambda(J/J^2)$, where $J = \ker \mu: A \otimes A \rightarrow A$ is the kernel of the multiplication mapping on A .

[Connes85] introduced a variant of the Hochschild homology in order to extend the Chern character to the algebraic K-theory of a C^* -algebra. For a field k of arbitrary characteristic, the cyclic homology $HC_*(A)$ of an algebra A over k is defined from a double complex in which each column is the Hochschild homology, and in each row we find the homology of a cyclic group. This leads to two spectral sequences ([Connes85]).

Theorem 12.4. *Given an algebra A over a field k , there are two spectral sequences converging to $HC_*(A)$. In the first spectral sequence, $E_{p,q}^1 \cong HH_{q-p}(A)$ and $d^1 = B$, the Connes boundary map. In the second, $E_{p,q}^1 \cong H_p(\mathbb{Z}/(q+1)\mathbb{Z}, A^{\otimes q+1})$, the homology of the group $\mathbb{Z}/(q+1)\mathbb{Z}$ with coefficients in the $\mathbb{Z}/(q+1)\mathbb{Z}$ -module $A^{\otimes q+1}$, where the generator of $\mathbb{Z}/(q+1)\mathbb{Z}$ acts on $A^{\otimes q+1}$ by the cyclic permutation $a_0 \otimes \cdots \otimes a_q \mapsto (-1)^q a_q \otimes a_0 \otimes \cdots \otimes a_{q-1}$.*

Cyclic homology figures in the computation of the algebraic K-theory of rings, in noncommutative differential geometry, and in mathematical physics. For a comprehensive and comprehensible survey of these ideas, see the excellent book of [Loday92].

Suppose we begin with a filtered augmented algebra, (A, μ, F, ε) , over a field, k and a filtered A -module, M , satisfying either of the following conditions:

$$I : \begin{cases} F_p A = A, F_p M = M, & \text{if } p \geq 0, \\ F_{-1} A = I(A) = \ker \varepsilon \quad \text{and} \quad \bigcap_p F_p A = \{0\} = \bigcap_p F_p M. \end{cases}$$

$$II : \begin{cases} F_p A = \{0\} = F_p M, & \text{if } p < 0, \\ F_0 A = k \quad \text{and} \quad \bigcup_p F_p A = A, \bigcup_p F_p M = M. \end{cases}$$

Theorem 12.5 (the May spectral sequence). *For a filtered k -algebra, A , and filtered A -module, M , satisfying I or II, there is a spectral sequence with $E^2 \cong \text{Tor}^{E^0 A}(k, E^0 M)$, and converging to $\text{Tor}_*^A(k, M)$. Dually, there is a spectral sequence with $E_2 \cong \text{Ext}_{E^0 A}(k, (E^0 M)^{\text{dual}})$, and converging to $\text{Ext}_A^*(k, M^{\text{dual}})$ as an $\text{Ext}_A^*(k, k)$ -module.*

When A and M are graded, the spectral sequences are trigraded, where the first two gradings sum to the homological degree and the last two sum to the internal degree. Applications of this spectral sequence to the case where A is a Hopf algebra were pioneered by [May64, 66]. A natural generalization of this spectral sequence is presented in exercise 3.4.

Other algebraic structures

The homological algebra of other algebraic structures, such as Hopf algebras, Lie algebras, and Leibniz algebras, was developed to obtain invariants that would aid in the classification problem of such structures and, more generally, lead to a deeper understanding of the structures themselves. The notion of an extension of Hopf algebras, Lie algebras, etc., plays the role of a fibration in topology, linking the members of the extension together. The case of a group extension, $1 \rightarrow K \rightarrow G \rightarrow Q \rightarrow 1$, is paradigmatic—the homological invariants of the constituent groups are linked together in the behavior of the Lyndon-Hochschild-Serre spectral sequence (Theorem 8^{bis}.12).

The analogue of the Lyndon-Hochschild-Serre spectral sequence for Lie algebras was introduced by [Hochschild-Serre53].

Theorem 12.6 (the Hochschild-Serre spectral sequence). *Let \mathfrak{h} be a Lie ideal in the Lie algebra \mathfrak{g} and M , a \mathfrak{g} -module. Then there is a spectral sequence, converging to $H^{p+q}(\mathfrak{g}, M)$, with $E_2^{p,q} \cong H^p(\mathfrak{g}/\mathfrak{h}, H_q(\mathfrak{h}, M))$.*

This theorem generalized results of [Koszul50] who worked with a relative version of cohomology for pairs of Lie algebras and over fields of characteristic zero, where the geometric theory of Lie groups provided motivation. [Koszul50] pioneered the homological algebra of Lie algebras as a tool independent of the topology and geometry involved.

A more general class of algebras that extends the notion of a Lie algebra was identified by [Loday93]: A **Leibniz algebra** \mathfrak{g} is k -module together with a bilinear mapping $[\ , \]: \mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{g}$ satisfying the Leibniz relation

$$[x, [y, z]] = [[x, y], z] - [[x, z], y].$$

The definition leaves out the antisymmetric relation expected of Lie algebras and so obtains a noncommutative version of a Lie algebra. A **Leibniz module** (or representation) is a k -module M together with a bilinear mapping $M \times \mathfrak{g} \rightarrow M$, written $(m, g) \mapsto [m, g]$, satisfying $[m, [x, y]] = [[m, x], y] - [[m, y], x]$.

[Loday-Pirishvili93] defined the Leibniz cohomology of a Leibniz algebra \mathfrak{g} with coefficients in a Leibniz module M , $HL^*(\mathfrak{g}, M)$, as the homology of the complex

$$\mathbb{R} \xrightarrow{0} C^1(\mathfrak{g}, M) \xrightarrow{d} C^2(\mathfrak{g}, M) \xrightarrow{d} \cdots \xrightarrow{d} C^k(\mathfrak{g}, M) \xrightarrow{d} \cdots$$

where $C^k(\mathfrak{g}, M) = \text{Hom}_{\mathbb{R}}(\mathfrak{g}^{\otimes k}, M)$ and the differential d is defined

$$\begin{aligned} d(\alpha)(g_1 \otimes \cdots \otimes g_{k+1}) = & \sum_{1 \leq i < j \leq k} (-1)^{j+1} \alpha(g_1 \otimes \cdots \otimes g_{i-1} \otimes [g_i, g_j] \otimes g_{i+1} \otimes \cdots \otimes \widehat{g}_j \otimes \cdots \otimes g_{k+1}) \\ & + [g_1, \alpha(g_2 \otimes \cdots \otimes g_{k+1})] + \sum_{i=2}^{k+1} (-1)^{i+1} [g_i, \alpha(g_1 \otimes \cdots \otimes \widehat{g}_i \otimes \cdots \otimes g_{k+1})]. \end{aligned}$$

The canonical mapping $\mathfrak{g}^{\otimes k} \rightarrow \mathfrak{g}^{\wedge k}$ induces a homomorphism $H_{\text{Lie}}^*(\mathfrak{g}, M) \rightarrow HL^*(\mathfrak{g}, M)$ when \mathfrak{g} is a Lie algebra. [Pirashvili94] and [Lodder98] have introduced a spectral sequence that computes the relative theory defined as

$$H_{\text{rel}}^*(\mathfrak{g}) = H(\Sigma^2 C^*(\mathfrak{g})/\Omega^*(\mathfrak{g}), \bar{d}),$$

where Σ is the operator that shifts degree, and $\Omega^*(\mathfrak{g})$ is the complex that defines Lie algebra cohomology. The spectral sequence measures the difference between the Lie algebra cohomology and the Leibniz cohomology when

they are applied to a Lie algebra, that is, it reveals the importance of the anticommutative condition on a Lie algebra. The E_2 -term of the spectral sequence is made up of the Leibniz cohomology of the Lie algebra \mathfrak{g} and another term defined as follows: There is a left \mathfrak{g} -module structure on $\mathfrak{g}^{\text{dual}}$ given by $g\gamma(h) = \gamma([h, g])$. We can define morphisms $i_1: \Omega^{n+1}(\mathfrak{g}) \rightarrow \Omega^n(\mathfrak{g}; \mathfrak{g}^{\text{dual}})$ and $i_2: \Omega^n(\mathfrak{g}; \mathfrak{g}^{\text{dual}}) \rightarrow C^{n+1}(\mathfrak{g}, \mathbb{R})$ by the formulas

$$\begin{aligned} i_1(\alpha)(g_1, g_2, \dots, g_n)(g_0) &= (-1)^n \alpha(g_0, g_1, \dots, g_n) \\ i_2(\beta)(g_0, g_1, \dots, g_n) &= (-1)^n \beta(g_1, \dots, g_n)(g_0). \end{aligned}$$

The complex $CR^*(\mathfrak{g})$ is defined by the short exact sequence

$$0 \rightarrow \Omega^*(\mathfrak{g}) \rightarrow \Omega^{*-1}(\mathfrak{g}, \mathfrak{g}^{\text{dual}}) \rightarrow \Sigma CR^*(\mathfrak{g}) \rightarrow 0.$$

The homology of $CR^*(\mathfrak{g})$ is denoted by $HR^*(\mathfrak{g})$.

Theorem 12.7. *Let \mathfrak{g} be a Lie algebra and M a \mathfrak{g} -module. Then there is a spectral sequence with $E_{p,q}^2 \cong HL^p(\mathfrak{g}, M) \otimes HR^q(\mathfrak{g})$, and converging to $H_{\text{rel}}^{p+q}(\mathfrak{g}, M)$.*

The spectral sequence was derived for Leibniz homology by [Pirashvili94] and for cohomology by [Lodder98], who has extended the Leibniz cohomology groups to diffeomorphism invariants of a manifold, and related them to the Gelfand-Fuks cohomology of smooth vector fields. He has also identified the Godbillon-Vey invariant of foliations as a Leibniz cohomology class.

The category of connected Hopf algebras over a ring R shares a great deal with the category of groups. One of the uses of group cohomology is the classification of extensions of groups. In particular, if $1 \rightarrow K \rightarrow G \rightarrow Q \rightarrow 1$ is an extension, then Q acts on K by conjugation, giving K a Q -module structure. There is also a twisting function $\tau: Q \times Q \rightarrow K$ and together the Q -module structure and τ determine the extension G up to a coboundary condition. This identifies $H^2(Q, K)$ as the group that classifies extensions with the given Q -module structure. [Gugenheim62] and [Singer72] carried out a similar development of the structure of an extension of connected Hopf algebras. The notion of a Q -module structure with twisting function is replaced with the notion of an *abelian matched pair* of Hopf algebras, (A, B) . The definition may be found in [Singer72, Definition 3.1]. This leads to cohomology groups $H^n(B, A)$, definable as a derived functors of an appropriate hom functor, or via a cotriple. [Henderson97] has studied the problem of computing the groups $H^n(B, A)$ for which he has introduced a spectral sequence.

Theorem 12.8. *If (A, B) is an abelian matched pair of graded connected Hopf algebras over a ring R , then there is a spectral sequence for each integer $r > 0$ with ${}_r E_2^{s,t} \cong \text{Ext}_B^{s,r}(R, \text{Cotor}_A^{t,r}(R, R))$ and converging to $E_1^{r,s+t-r}$, the E_1 -term of a spectral sequence that converges to $H^*(B, A)$.*

The method of construction is to use cosimplicial objects and interpret the various filtration quotients. [Henderson97] computed the spectral sequences in the case of a truncated monogenic tensor algebra, from which he determined the nature of certain extensions over \mathbb{F}_p that occur in the study of finite H-spaces ([Lin, J78]).

Abelian categories

The structure of a spectral sequence requires a homological algebra that includes filtrations, subquotients, and additivity of morphisms. The minimal requirements of a category in which spectral sequences may be constructed and studied were identified by [Grothendieck57], who introduced the notion of an abelian category. General results about spectral sequences in abelian categories soon followed in papers of [Dold62], [Eilenberg-Moore62], and [Eckmann-Hilton66]. For a thorough introduction to the homological algebra of abelian categories, see the book of [Tamme94].

Among the most important results of [Grothendieck57] is a general spectral sequence whose instances include many classical results. We begin with abelian categories, **AbelCat**₁, **AbelCat**₂, and **AbelCat**₃ along with functors

$$F: \mathbf{AbelCat}_1 \rightarrow \mathbf{AbelCat}_2 \quad \text{and} \quad G: \mathbf{AbelCat}_2 \rightarrow \mathbf{AbelCat}_3.$$

We relate the derived functors F and G to the derived functors of $(G \circ F)$.

Theorem 12.9 (the composite functor spectral sequence). *Suppose the functors F and G are covariant, G is left exact and F takes injective objects in **AbelCat**₁ to G -acyclic objects in **AbelCat**₂ (G -acyclic objects have the property that the derived functors of G vanish on them). Then there is a spectral sequence with*

$$E_2^{p,q} \cong (R^p G)(R^q F(A)),$$

and converging to $R^(G \circ F)(A)$ for A in **AbelCat**₁.*

The homological invariants we have considered are instances of the derived functors of such functors as $M \otimes_{\Gamma} -$ or $\text{Hom}_{\Gamma}(-, C)$. The interested reader can review the spectral sequences of §12.1, §2.4, §7.1, and §9.2 and try to derive these spectral sequences as instances of the composite functor spectral sequence.

When the categories involved are not abelian, it is still possible to set up a Grothendieck spectral sequence. Using simplicial methods and homotopy

theory, [Blanc-Stover92] have generalized the composite functor spectral sequence to categories of universal algebras (such as groups, rings, Lie algebras, etc.) and more general functors.

An example of the Grothendieck spectral sequence is a generalization of the change-of-rings spectral sequence. Suppose we have a family of functors, indexed over \mathbb{Z} ,

$$\{T_n\}: \mathbf{AbelCat}_1 \rightarrow \mathbf{AbelCat}_2$$

that act like the derived functors of T_0 . That is, they are additive, left or right exact, and to a short exact sequence in $\mathbf{AbelCat}_1$, $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$, there is a long exact sequence in $\mathbf{AbelCat}_2$

$$\cdots \rightarrow T_n A \rightarrow T_n B \rightarrow T_n C \rightarrow T_{n\pm 1} A \rightarrow \cdots$$

(where ± 1 depends on the variance of the T 's).

Theorem 12.10 (the Universal Coefficient spectral sequence). *Suppose A is an object in $\mathbf{AbelCat}_1$ and $M \in {}_A\mathbf{Mod}$*

- (1) *projdim $M < \infty$ or $T_{-N} = 0$ for sufficiently large N ,*
- (2) *A is Noetherian and M is finitely generated or T_q , for $q \leq n$, T_q commutes with arbitrary direct sums.*

Then there is a spectral sequence with $E_{p,q}^2 \cong \mathrm{Tor}_p^A(T_q A, M)$, and converging to $T_ M$. Dually, for $T^* = T_{-*}$, contravariant, there is a spectral sequence with $E_2^{p,q} \cong \mathrm{Ext}_A^p(M, T^q A)$, and converging to $T^* M$.*

This theorem is proved in this generality in the paper of [Dold62]. For the abelian categories of chain and cochain complexes, the familiar Universal Coefficient theorem can be recovered. For $T_n = \mathrm{Ext}_A^{-n}(B, -)$, this gives another spectral sequence relating the homological invariants of rings and modules.

Another application may be made to compute the **hypercohomology** of a complex, (A^*, d) , of objects in $\mathbf{AbelCat}_1$ with respect to a left exact functor $F: \mathbf{AbelCat}_1 \rightarrow \mathbf{AbelCat}_2$. Suppose $\mathbf{AbelCat}_1$ has enough injectives. Then the **hyperderived** functors of F can be defined: Suppose (I^\bullet, ∂) is a complex of injective objects in $\mathbf{AbelCat}_1$ with $H(I^\bullet, \partial) \cong H(A^*, d)$. The hypercohomology of A^* is defined by $\mathbb{H}F^*(A^*) = H(FI^\bullet, F\partial)$. This definition can be shown to be independent of the choice of injective object.

Theorem 12.11 (the hypercohomology spectral sequence). *If $\{R^j F\}$ denotes the sequence of right derived functors of F , then there is a spectral sequence, with $E_2^{p,q} \cong (R^p F)(H^q(A^*, d))$, and converging to $\mathbb{H}F^*(A^*)$.*

This theorem can be proved from the composite functor spectral sequence. A direct proof for rings and modules appears in the classic books of [Cartan-Eilenberg56] and [Mac Lane75].

12.2 Spectral sequences in Geometry

The basic objects that are studied in algebraic geometry, varieties and schemes, carry many different structures. Similarly, the basic objects in differential geometry, manifolds, are rich with structure. There is an underlying topological space (sometimes with the nonHausdorff Zariski topology), possible analytic structure, and, for varieties, the underlying structure of polynomial rings; the interaction between these structures and with the homological invariants of such objects leads to many useful spectral sequences.

The first example is historically the first spectral sequence. [Leray46], in a series of *Comptes Rendues* notes introduced the notions of a sheaf over a topological space, cohomology with coefficients in a sheaf, and the Leray spectral sequence associated to a mapping (§6.4). Suppose X and Y are spaces with $f: X \rightarrow Y$, a continuous mapping. Suppose $\Phi = \{U_\alpha \mid \alpha \in J\}$ is an open cover of Y . Such a covering gives rise to a left exact functor from the category of sheaves on X to the category of sheaves on Y , which is constructed from the presheaf of sections. For a sheaf S on X ,

$$\Gamma_\Phi(S) = \text{sheaf derived from the presheaf } U_\alpha \mapsto \{\Gamma(f^{-1}(U_\alpha), S) \mid U_\alpha \in \Phi\}.$$

The theorem of [Leray46] relates the sheaves, S on X and $\Gamma_\Phi(S)$ on Y . The category of sheaves (of abelian groups) is an abelian category and so there is a notion of homological algebra for sheaves.

Theorem 12.12 (the Leray spectral sequence). *Let $R^*\Gamma_\Phi$ denote the right derived functors of Γ_Φ . If \check{H}^* denotes the sheaf cohomology of a space in a given sheaf, then there is a spectral sequence with $E_2^{p,q} \cong \check{H}^p(Y, R^q\Gamma_\Phi(S))$, and converging to $\check{H}^*(X, S)$.*

An application of the Leray spectral sequence is the case of a complex variety. Algebraically, the variety has the Zariski topology. Analytically, it carries a topological manifold structure. The sheaves of germs of functions on the variety (analytic and algebraic) and the cohomology of the variety in these sheaves are related by the spectral sequence in the theorem and the continuous function $X_{\text{Zar}} \rightarrow X_{\mathbb{C}}$. This example also reveals a role played by spectral sequences in algebraic geometry—patching local data into global data. Another example of a spectral sequence focusing on patching is the **local-to-global** spectral sequence.

Theorem 12.13. *Suppose X is a topological space and \mathcal{S} , a sheaf of rings on X . Suppose that \mathcal{M} and \mathcal{N} are sheaves of left \mathcal{S} -modules. Then there is a spectral sequence with $E_2^{p,q} \cong H^p(X, \mathcal{E}xt_{\mathcal{S}}^q(\mathcal{M}, \mathcal{N}))$, and converging to $\text{Ext}_{\mathcal{S}}^*(\mathcal{M}, \mathcal{N})$, which denotes the derived functors of*

$$\text{Hom}_{\mathcal{S}}(\mathcal{M}, \mathcal{N}) = \prod_{x \in X} \text{Hom}_{\mathcal{S}(x)}(\mathcal{M}(x), \mathcal{N}(x)).$$

We refer the reader to another classic text on spectral sequences, [Godement58], for a discussion of local-to-global spectral sequences and a proof of this theorem.

Spectral sequences and the de Rham complex

Other sources of spectral sequences in algebraic and differential geometry are filtrations of the de Rham complex that derive from some structural feature of the particular situation. As a first example, we mention a spectral sequence introduced by [Grothendieck61].

If X is a nonsingular algebraic variety of dimension n over a field k of arbitrary characteristic, then define

$$\Omega_{X/k}^1 = \text{the sheaf of differential 1-forms on } X \text{ over } k.$$

We form the exterior algebra, $\Omega_{X/k}^i = \Lambda^i \Omega_{X/k}^1$. Then there is a natural derivation $d: \mathcal{O}_X \rightarrow \Omega_{X/k}^1$ and so an exterior derivative giving a complex

$$\mathcal{O}_X \xrightarrow{d} \Omega_{X/k}^1 \xrightarrow{d} \cdots \xrightarrow{d} \Omega_{X/k}^{n-1} \xrightarrow{d} \Omega_{X/k}^n,$$

called the **algebraic de Rham complex** on X . The hypercohomology of this complex is called the **algebraic de Rham cohomology** of X , and denoted by $H_{\text{deR}}^*(X)$.

Theorem 12.14 (the Hodge-de Rham spectral sequence). *There is a spectral sequence with $E_1^{p,q} \cong H^p(X, \Omega_{X/k}^q)$, the cohomology of X in the sheaf $\Omega_{X/k}^q$, and converging to $H_{\text{deR}}^*(X)$.*

[Grothendieck61] related the algebraic de Rham cohomology of a finite dimensional variety, X , over \mathbb{C} to singular cohomology by using the spectral sequence to prove

$$H_{\text{deR}}^*(X) \cong H_{\text{sing}}^*(X^{\text{an}}; \mathbb{C})$$

where X^{an} is the analytic space associated to X . The filtration on $H_{\text{deR}}^*(X)$ derived from this spectral sequence is related to the system of weights due to the Hodge structure on a compact complex variety. This relation has been studied thoroughly by [Deligne71].

When X is a scheme, smooth and proper over a perfect field k of characteristic $p > 0$, then there are other invariants that reflect the p -adic structure of the scheme. In particular, there is the crystalline cohomology of X , $H_{\text{cris}}^*(X/W)$, where W denotes the ring of Witt vectors, defined by [Grothendieck68] and [Bertholet76]. [Bloch78] studied the relations between the various cohomological invariants of a variety over a perfect field of characteristic $p > 2$. He introduced a spectral sequence to compute $H_{\text{cris}}^*(X/W)$ using a complex C_X^\bullet

of typical curves on K-theory. The hypercohomology spectral sequence in this case is called the **slope spectral sequence** and has $E_1^{p,q} \cong H^q(X, C_X^p)$ and converges to $H_{\text{cris}}^*(X/W)$. By analogy with the Hodge-de Rham spectral sequence, Deligne ([Illusie79]) introduced the *de Rham-Witt complex* for a scheme X , $W\Omega_X^\bullet$, which agrees with C_X^\bullet when the latter is defined.

Theorem 12.15 (the slope spectral sequence). *Let X be a scheme, smooth and proper over a perfect field k of characteristic $p > 0$ and $W\Omega_X^\bullet$ the associated de Rham-Witt complex. Then there is a spectral sequence with $E_1^{p,q} \cong H^q(X, W\Omega_X^p)$, and converging to $H_{\text{cris}}^*(X/W)$.*

[Ekedahl86] has written a booklength account of the structure theory of the slope spectral sequence. The applications of this spectral sequence are numerous in algebraic geometry ([Illusie79], [Ekedahl86]).

Suppose M is a finite dimensional complex manifold. The cotangent bundle T^*M of M admits a decomposition, $T^*M = T^{*'}M \oplus T^{*''}M$, into holomorphic forms (sums $\sum f_i dz_i$ with f_i a holomorphic function on M) and antiholomorphic forms (sums $\sum F_i d\bar{z}_i$ with \bar{F}_i holomorphic). This decomposition induces a bigrading on the de Rham complex of \mathbb{C} -valued differential forms on M , $\Omega^*(M, \mathbb{C})$, where $\Omega^{p,q}(M, \mathbb{C}) = \Lambda^p(T^{*'}M) \otimes \Lambda^q(T^{*''}M)$. An n -form with $n = p + q$ in $\Omega^{p,q}(M, \mathbb{C})$ is called a (p, q) -**form**. The exterior differential on $\Omega^*(M, \mathbb{C})$ takes a form $\omega \in \Omega^{p,q}(M, \mathbb{C})$ to the direct sum $\Omega^{p+1,q}(M, \mathbb{C}) \oplus \Omega^{p,q+1}(M, \mathbb{C})$. Composing with the projections we get the expression $d = \partial + \bar{\partial}$ with ∂ of bidegree $(1, 0)$ and $\bar{\partial}$ of bidegree $(0, 1)$. Furthermore, $\partial \circ \partial = 0 = \bar{\partial} \circ \bar{\partial}$.

It follows that the data $(\Omega^{p,q}(M, \mathbb{C}), \partial, \bar{\partial})$ determine a double complex whose total complex is the de Rham complex. The vertical differential, $\bar{\partial}$, leads to the **Dolbeault cohomology of M** ,

$$H_{\bar{\partial}}^{p,q}(M) = \Omega^{p,q}(M, \mathbb{C}) \cap \ker \bar{\partial} / \bar{\partial}(\Omega^{p,q-1}(M, \mathbb{C})).$$

Theorem 12.16 (the Frölicher spectral sequence). *Given a complex manifold M , there is a spectral sequence converging strongly to $H_{\text{deR}}^*(M, \mathbb{C})$ with E_2 -term given by $E_2^{p,q} \cong H_{\bar{\partial}}^{p,q}(M)$.*

[Frölicher55] introduced the spectral sequence to relate the geometric invariants of the Dolbeault complex to the topological invariants of the de Rham cohomology. He observed that a complex manifold with a positive definite Kähler metric has $E_1 \cong E_\infty$, and so the spectral sequence gives a necessary condition for the existence of a Kähler structure. [Cordero-Fernández-Gray91, 93] have given examples of complex manifolds for which the spectral sequence does not collapse at E_1 .

When X is a smooth projective complex variety, there is also a Hodge filtration on the cohomology of X . Because it has an underlying Kähler manifold, the Frölicher spectral sequence collapses for X . In the more general case of a quasi-projective complex variety V (that is, $V = X - Y$ for X and Y complex varieties), [Deligne68] has proved that there is a different filtration on $H^*(X; \mathbb{Q})$, called a **weight filtration**. Such a filtration is increasing

$$0 = W_{-1} \subset W_0 \subset W_1 \subset \cdots \subset W_{2m} = H^m(X; \mathbb{Q})$$

for which the complexified associated graded group $E_l^0 \otimes \mathbb{C} = (W_l/W_{l-1}) \otimes \mathbb{C}$ has a decomposition of Hodge form $E_l^0 = \bigoplus_{p+q=l} H^{p,q}$. Such a structure is called a **mixed Hodge structure**. [Deligne71] proved that if X is a quasiprojective algebraic variety that there is a weight filtration on $H^m(X; \mathbb{Q})$ and a decreasing Hodge filtration on $H^m(X; \mathbb{C})$ such that the filtration induced by the Hodge filtration on the complexified associated graded module from the weight filtration was a mixed Hodge structure. The existence of two filtrations of this sort can lead to the collapse of the associated spectral sequences as proved by [Deligne71]. For an introduction to mixed Hodge structures, see the ‘naive guide’ of [Durfee83]. The appearance and uses of spectral sequences from mixed Hodge structures is developed in the book of [El Zein91].

If (M, g) is an n -dimensional Riemannian manifold and $\pi: E \rightarrow M$ is a flat vector bundle, then a smooth distribution of k -planes $A \subset TM$ together with its orthogonal complement B leads to a decomposition of the metric $g = g_A \oplus g_B$. If we vary the metric by $g_\delta = g_A + \delta^{-2}g_B$ for $0 < \delta \leq 1$, then we obtain a family of Laplacians for (M, g_δ) and a corresponding exterior derivative on $\Omega^*(M; E)$. There is a filtration on the L^2 -completion of $\Omega^p(M; E)$ given by $\omega \in F^k$ when there is a j with $d_\delta(\omega + \delta\omega + \cdots + \delta^j\omega) \in \delta^k\Omega^{p+1}[\delta]$. A spectral sequence results that has been shown to be isomorphic to the Leray spectral sequence associated to the splitting $A \otimes B = TM$ ([Mazzeo-Melrose90], [Forman95]). This spectral sequence is related to the behavior of the spectrum of the Laplacians involved and is called the **adiabatic spectral sequence**. For a general discussion of these ideas, see the paper of [Forman94].

The bigrading of the de Rham complex in Hodge theory has a striking relative in the calculus of variations. Here one wants to study differential equations as sections of jet bundles associated to a smooth vector bundle $\pi: E \rightarrow M$. Let $J^\infty(E) \rightarrow M$ denote the infinite order jet bundle associated to π . Let I denote the *contact ideal*, the differential ideal of the de Rham complex $\Omega^*(J^\infty(E))$ of forms that pull back to zero under any extension to infinite jets of a section $s: M \rightarrow E$. A bigrading results on $\Omega^*(J^\infty(E))$ by counting the number of forms from I needed to express a given form. The exterior derivative can be decomposed into horizontal and vertical components giving a double complex, known as the **variational bicomplex**. The associated spectral sequence was identified by [Vinogradov78] and a clear presentation can be found in the monograph of [Krasil'shchik-Verbotevsky98].

Theorem 12.17 (the \mathcal{C} -spectral sequence). *Let $\pi: E \rightarrow M$ be a smooth vector bundle over an n -dimensional manifold M . Then the spectral sequence associated to the variational bicomplex converges to $H_{\text{deR}}^*(J^\infty(E))$ and has $E_1^{0,q}$ isomorphic to the horizontal cohomology associated to π ; $E_1^{p,n}$ isomorphic to the modules $L_p^{\text{alt}}(\pi)$ for $p > 0$, where L_p^{alt} is the homology of the complex of alternating differential operators associated to π , and $E_1^{r,s} \cong \{0\}$, otherwise. The differential $d_1: E_1^{0,n} \rightarrow E_1^{1,n}$ can be identified with the operator that associates to a Lagrangian its Euler-Lagrange equation.*

In the case of a specific distribution on a subspace of $J^\infty(E)$, the horizontal cohomology is based on the associated Cartan submodule determined by the equation (hence the \mathcal{C} -spectral sequence) and the E_1 -term is more complicated to describe. Applications of the variational bicomplex are presented by [Anderson-Thompson92]; applications of the \mathcal{C} -spectral sequence by [Krasil'shchik98].

Finally, to close this section we mention work of [Dixon91] on the computation of BRS cohomology for gauge systems ([Henneaux-Teitelboim92]). The BRS operator determines a differential on the Fock space of integrated local polynomial functions of a Yang-Mills field and a Fadeev-Popov ghost field. The resulting cohomology determines invariants of a gauge system, such as the ghost numbers, the Lorentz character, and discrete symmetries. [Dixon91] filtered the space on which the BRS operator acts and deduced the associated spectral sequence. The induced grading from the E_∞ -term of the spectral sequence presents the desired complicated cohomology in simpler pieces that are computable.

12.3 Spectral sequences in algebraic K-theory

Algebraic K-theory defines a sequence of invariants for a ring R and these invariants may be constructed as the homotopy groups of a certain space (or a certain spectrum). The tools for the study of algebraic K-theory are as varied as the appearances of rings throughout mathematics and so there are many structures in concert in algebraic K-theory.

To a ring R we associate the scheme $\text{Spec } R$ with the Zariski topology. [Brown, K-Gersten73] and [Quillen73] derived a spectral sequence defined for cohomology groups related to simplicial sheaves and applicable to algebraic K-theory: Suppose X is a Noetherian space (that is, the open sets in X satisfy the ascending chain condition) and suppose that the irreducible closed subsets of X also satisfy the ACC (for example, if $X = \text{Spec } R$ for R a regular, commutative ring). A **simplicial sheaf** on X is a sheaf with values in **SimpEns**, the category of simplicial sets. If K is a simplicial sheaf on X , then we say that K is **flasque** if the mapping $t: K \rightarrow *$ satisfies the property that, for U, V open in X ,

$$\Gamma(V, K) \xrightarrow{(\Gamma(V,t), \text{restr})} \Gamma(V, *) \otimes_{\Gamma(U,*)} \Gamma(U, K)$$

is a simplicial fibration.

Replace the functors $\Gamma(U, -)$, for U open in X , by a functor $R\Gamma(U, -)$ defined on the *homotopy* category of sheaves over X ,

$$R\Gamma(U, -): \mathbf{HoSimpSheaves}_X \rightarrow \mathbf{HoSimpEns}$$

such that, if K is flasque, then there is natural isomorphism of $R\Gamma(U, K)$ with $\Gamma(U, K)$. Define the **generalized sheaf cohomology groups**, $H^*(X, K)$ by

$$H^q(X, K) = \pi_{-q}(R\Gamma(X, K)).$$

Theorem 12.18 (the Brown-Gersten-Quillen spectral sequence). *Suppose X is a Noetherian space of finite Krull dimension and K a simplicial sheaf with basepoint that satisfies $\pi_0(K) = \{0\}$ and $\pi_1(K)$ and $H^{-1}(X, K)$ are abelian. Suppose $H^p(X, \pi_n(K)) = \{0\}$ for $p \geq n$. Then there is a fourth quadrant cohomological spectral sequence with $E_2^{p,q} \cong H^p(X, \pi_{-q}(K))$, and converging to $H^*(X, K)$.*

If $X = \text{Spec } R$ for R regular and such that every coherent sheaf on X is quotient of a locally free sheaf, then one builds a simplicial sheaf on X from the classifying construction for a category with $\pi_{-q}(K) = \mathcal{K}_{-q}$, the abelian sheaf of local K -groups of R and satisfying, $H^*(X, K) = K_*(R)$. In this case,

$$E_2^{p,q} \cong H^p(\text{Spec } R, \mathcal{K}_{-q})$$

and the spectral sequence converges to $K_*(R)$. A construction and discussion of the applications of the Brown-Gersten-Quillen spectral sequence may be found in the book of [Srinivas96].

Another example of an invariant of a scheme (a ringed space) is its étale cohomology ([Tamme94]). In this we relate the étale cohomology of a scheme with coefficients in various cyclic groups, to the localized algebraic K -theory of the scheme.

In order to introduce $\mathbb{Z}/m\mathbb{Z}$ coefficients on homotopy groups of a spectrum, one smashes the spectrum with the appropriate Moore spectrum for $\mathbb{Z}/m\mathbb{Z}$ ([Browder78]). The algebraic K -theory of a scheme with coefficients in $\mathbb{Z}/m\mathbb{Z}$ can be defined analogously by taking the spectrum associated to the scheme and smashing it with the Moore spectrum; its homotopy groups are denoted by $(K/m)_*(X)$.

Suppose $b \in (K/m)_*(X)$ and we consider the direct limit of the system

$$(K/m)_*(X) \xrightarrow{b \times -} (K/m)_*(X) \xrightarrow{b \times -} \dots$$

given by left multiplication by b . This direct limit is called the **localization** of $(K/m)_*(X)$ with respect to b (or by inverting the element b), and it is denoted by $(K/m)_*(X)[b^{-1}]$.

In connection with the Lichtenbaum-Quillen conjecture, [Thomason85] introduced a descent spectral sequence associated to schemes X satisfying certain technical conditions:

Theorem 12.19 (the descent spectral sequence). *Suppose l is a fixed prime and ν , a natural number. Let X be a separated, Noetherian, regular scheme of finite Krull dimension, with sufficiently nice residue fields of characteristic different from l . Suppose β is the Bott element in $(K/l^\nu)_2(X)$, that is, the element such that the Bockstein of β is an appropriate power of an l^{th} root of unity in $K_1(X)$. Then there is a spectral sequence with differentials of bidegree $(r, r - 1)$,*

$$E_2^{p,q} \cong \begin{cases} H_{\text{et}}^p(X; \mathbb{Z}_l(i)), & \text{if } q = 2i, \\ \{0\}, & \text{if } q, \text{ odd,} \end{cases}$$

where $\mathbb{Z}_l(i) = \mathbb{Z}_l(1)^{\otimes i}$ is the i^{th} Tate twist of the l -adic integers. The spectral sequence converges to $(K/l^\nu)_*(X)[\beta^{-1}]$.

The Quillen-Lichtenbaum conjecture relates the order of the K-groups to values of zeta functions for certain arithmetic number fields. [Thomason85] proved it for this localized version of algebraic K-theory. The result also applies to the case of X , a variety over an algebraically closed field k of characteristic $\neq l$ and so allows computation of these algebraic K-groups for such varieties. [Mitchell97] presented a proof of Thomason's theorem in terms of hypercohomology spectra in which he exposes many of the details and conceptual underpinnings of this result, as well as the applications. [Thomason82, 83] described a context where this theorem is a case of a homotopy limit problem, here for diagrams of spectra (see §11.4).

The next spectral sequence has played a key role in recent developments of Voevodsky in his proof of the the Quillen-Lichtenbaum conjecture at 2 for fields of characteristic zero ([Friedlander97]). An important tool in algebraic K-theory is the *motivic cohomology* of a field. Motivic cohomology is a functor on schemes that plays the role of singular cohomology for spaces. In particular, the Atiyah-Hirzebruch spectral sequence has $E_2^{p,q} \cong H^p(X; K_{\text{top}}^q)$ and converges to $K_{\text{top}}^{p+q}(X)$. Thus (following Beilinson), the motivic cohomology of the scheme with coefficients in the algebraic K-theory of a point ($\text{Spec}(k)$) would converge to the algebraic K-theory of the scheme. Furthermore, tensored with the rational numbers, this spectral sequence would collapse determining the algebraic K-theory groups mod torsion. [Bloch86] has proposed that motivic cohomology of $\text{Spec}(k)$, $H_{\mathcal{M}}^r(\text{Spec}(k), \mathbb{Z}(s))$, may be identified with the higher Chow groups $CH^s(\text{Spec } k, 2s - r)$ and with this definition, there is a spectral sequence derived by [Bloch-Lichtenbaum94].

Theorem 12.20 (the Bloch-Lichtenbaum spectral sequence). *Let k denote a field. There is a fourth quadrant spectral sequence with*

$$E_2^{p,q} \cong H_{\mathcal{M}}^{p-q}(\text{Spec}(k), \mathbb{Z}(-q)),$$

and converging to $K_{-p-q}(\text{Spec } k)$.

Veodovsky used his proof of the Milnor conjecture ([Kahn, B97], [Morel98]), together with this spectral sequence to obtain his proof of the Quillen-Lichtenbaum conjecture.

Finally, we close this section with another analogue of the Atiyah-Hirzebruch spectral sequence, this time for a different K-theory and a different filtration. If A is a C^* -algebra, then there is a K-theory of A , defined and developed by [Brown-Douglas-Fillmore77], [Pimsner-Popa-Voiculescu79], and [Kasparov79]. [Schochet81] introduced a spectral sequence that applies when A is a filtered C^* -algebra, that is, there is a sequence of closed ideals,

$$A_0 \subset A_1 \subset \cdots \subset A_n \subset A_{n+1} \subset \cdots \subset A,$$

and $A = \text{cls}(\bigcup_n A_n)$.

Theorem 12.21. *Suppose given a filtered C^* -algebra, $(A, \{A_n\})$. Then there is a spectral sequence with $E_{p,q}^1 \cong K_{p+q}(A_p/A_{p-1})$, and converging to $K_*(A)$. The spectral sequence is natural with respect to filtration-preserving maps of C^* -algebras.*

This result and other results of Schochet bring the technical tools of algebraic topology to bear on the study of C^* -algebras.

12.4 Derived categories

The functors of homological algebra such as Tor and Ext are defined as the homology of chain complexes that are built in a noncanonical manner. In order to obtain homological invariants, the chain complexes must be carefully chosen. For example, a projective resolution of a right A -module,

$$\cdots \rightarrow P^{-i+1} \rightarrow P^{-i} \rightarrow \cdots \rightarrow P^{-1} \rightarrow P^0 \rightarrow M \rightarrow 0,$$

gives $\text{Tor}_*^A(M, N)$ by computing $H(P^\bullet \otimes_A N)$ for a left A -module N . Other choices of projective resolution can be compared with this particular choice to give isomorphic Tor groups, that is, groups that depend on A , M , and N only. In the case of modules, flat modules have the property of exactness on tensoring over A and so the axiomatic properties of Tor can be achieved by a choice of a flat resolution. However, it may be difficult to compare two flat resolutions.

Grothendieck and [Verdier63/97] defined the notion of the derived category of an abelian category \mathbf{A} in an effort to establish a framework in which to extend the duality results of [Serre54]. Let $\mathbb{C}(\mathbf{A})$ denote the category of chain complexes of objects and degree zero maps of complexes in \mathbf{A} . Let $\mathbb{C}^+(\mathbf{A})$ ($\mathbb{C}^-(\mathbf{A})$) denote the subcategory of chain complexes that are bounded below (above). A morphism of complexes $P^\bullet \rightarrow Q^\bullet$ is a **quasi-isomorphism** if it induces an isomorphism $H(P^*) \rightarrow H(Q^*)$ of graded objects. The **derived**

category of \mathbf{A} , $\mathbb{D}(\mathbf{A})$, is obtained by formally inverting the class of quasi-isomorphisms in $\mathbb{C}(\mathbf{A})$. This formal inversion can be made concrete by using a calculus of fractions developed by [Verdier63/97] and [Gabriel-Zisman67].

If $F: \mathbf{A} \rightarrow \mathbf{B}$ is an additive functor between abelian categories, then we can ask if there is an extension of F to a functor $\mathbb{D}^+(\mathbf{A}) \rightarrow \mathbb{D}^+(\mathbf{B})$. A minimal requirement is that F , extended levelwise to $\mathbb{C}^+(\mathbf{A}) \rightarrow \mathbb{C}^+(\mathbf{B})$, preserve quasi-isomorphisms. This is true if F is left exact. The right derived functors of F determine the extension of F to $RF: \mathbb{D}^+(\mathbf{A}) \rightarrow \mathbb{D}^+(\mathbf{B})$. This extension is proved to exist by analyzing the mapping cylinder construction in abelian categories, a construction formalized in the notion of a *triangulated category*.

When the abelian category \mathbf{A} has enough injectives, then the value of $H^s(RF(K^\bullet))$ is called the s^{th} **hyperderived functor** of F with respect to the complex K^\bullet . The computation of $H^*(RF(K^\bullet))$ may be carried out by replacing K^\bullet with a double complex of injective objects, from which there is a spectral sequence with $E_2^{p,q} \cong (R^p F)(H^q(K^\bullet))$, converging weakly to $H^*(RF(K^\bullet))$.

The point of derived categories, however, is to argue with the objects up to equivalence and the derived functors as functors on a particular category. An example of this principle in action is the following basic result.

Theorem 12.22. *Given three abelian categories \mathbf{A} , \mathbf{B} , and \mathbf{C} , and additive left exact functors $F: \mathbf{A} \rightarrow \mathbf{B}$ and $G: \mathbf{B} \rightarrow \mathbf{C}$ such that F takes injective objects in \mathbf{A} to G -acyclic objects in \mathbf{B} , then the extensions of F , G and $G \circ F$ to the derived categories are naturally isomorphic, that is, $R(G \circ F) \cong R(G) \circ R(F)$.*

(Proofs of this theorem can be found in the book of [Weibel94, 10.8.2], or [Gelfand-Manin96, III.7.1] or in the survey paper of [Keller96].) When the spectral sequence is applied to compute the hyperderived functors of the product, we recover the Grothendieck spectral sequence (Theorem 12.9). The underlying equivalence is more revealing than the spectral sequence and the derived category provides the framework to make such insights.

The language of derived categories is based on the basic structures of stable homotopy theory. [May94] has given a dictionary between algebra and topology that illuminates the analogies. The homological algebra of rings and modules can be carried back to stable homotopy through the foundational work of [Elmendorf-Kriz-Mandell-May97].

Derived categories have spread throughout mathematics wherever homological algebra has developed. As derived categories provide organization, spectral sequences will provide computations.